ATR_0 **and Some Related Theories**

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Contents

Introduction **1**

I. Subsystems of Second Order Arithmetic **3**

1. Preliminaries **5**
1.1. The language $\mathcal{L}_2$ 5
1.2. Semantics of $\mathcal{L}_2$ 11
1.3. The system ACA_0 12
1.4. Mathematical notions within ACA_0 14

2. ATR_0 and relatives **17**
2.1. The system ATR_0 17
2.2. Relatives of ATR_0 18
2.3. Equivalence proofs 20

3. ATR_0 without set-parameters **31**
3.1. The systems ATR_0^- and ATR^- 31
3.2. Proof-theoretic strength of a theory 34
3.3. The systems FP_0^- and $\mathsf{pr\text{-}ATR}_0^-$ 36
3.4. The systems FP^- and $\mathsf{pr\text{-}ATR}^-$ 44

II. Subsystems of Set Theory **47**

4. Preliminaries **49**
4.1. The language $\mathcal{L}_s$ 49
4.2. Basic set theory 53
4.3. Objects as trees 57

5. Interpreting BS, BS^1 and BS^0 in Second Order Arithmetic **69**
5.1. Translating from $\mathcal{L}_s$ to $\mathcal{L}_2$ 69
5.2. Interpreting (RC) and co. 74
5.3. Interpreting (Δ_0-$\mathrm{I_N}$), ($\mathcal{L}_s$-$\mathrm{I_N}$) and ($\mathcal{L}_s$-$\mathrm{I}_\in$) 83

6. Adding Σ and Π reduction **87**
6.1. Π and Σ reduction 87
6.2. Results 91

7. Adding Axiom Beta **93**
7.1. The systems BETA_0 and BETA 93
7.2. Relation to $\mathsf{ATR}_0^{\mathsf{set}}$ and $\mathsf{ATR}^{\mathsf{set}}$ 96
7.3. Adding Σ and Π reduction 102
7.4. Moving to Kripke-Platek set theory 104

A. Appendix **107**

Bibliography **115**

Index **119**

Introduction

The main point of interest of this dissertation is to study theories related to the theory ATR_0 in the realm of second order arithmetic and set theory. Second order arithmetic constitutes of a collection of axiomatic systems that formalises the natural numbers and their subsets. It turns out that a big branch of ordinary mathematics can be formalised within five subsystems of increasing strength, which are sometimes denoted the Big Five:

$$\mathsf{RCA}_0 \subsetneq \mathsf{WKL}_0 \subseteq \mathsf{ACA}_0 \subsetneq \mathsf{ATR}_0 \subsetneq \Pi^1_1\text{-}\mathsf{CA}_0.$$

It can be shown that many mathematical theorems are equivalent to one of the Big Five's defining set existence axiom over some base theory, usually RCA_0. Research in that direction is summarised under the programme of "Reverse mathematics" by Friedman/ Simpson. The textbook by Simpson, cf. [Sim09], provides an extensive exploration and overview of this subject and is referred to frequently. The defining set existence axiom of ATR_0 is arithmetical transfinite recursion, denoted (ATR). ATR_0 has proof-theoretic strength Γ_0. Feferman and Schütte characterised Γ_0 as the limit of predicative mathematics, cf. [Fef64, Sch77]. An overview on the topic of predicativity can be found in [Fef05]. Moreover, ATR_0 corresponds to the foundational program of predicative reductionism, cf. [Sim85, Sim88].

In Part I we choose ACA_0 as our base theory. In chapter 2 we then prove the equivalence of several axiom schemas to (ATR) over ACA_0, therefore rounding up the general picture. The following axiom schemas are considered: The schema (FP) introducing fixed points for positive arithmetical operators; a more general fixed point principle, denoted (MΔ^1_1-FP), prescribing fixed points for monotone Δ^1_1 operators; the schema (w-Σ^1_1-TDC), which is short for weak Σ^1_1 dependent choice; the reduction principles (Σ^1_1-Red) and (Π^1_1-Red), denoted as, respectively, Π^1_1 and Σ^1_1 separation in [Sim09]; and finally, a transfinite recursion principle featuring the iteration of Δ^1_1 operators along well orderings, denoted (Δ^1_1-TR). (Π^1_1-Red) is of special interest since it has very natural set-theoretic counterparts. This leads to interesting questions, which are discussed in Part II. Note that in order to comply with standard notations of set theory, we deviate from [Sim09] and speak of reduction principles rather than separation principles.

Part I is concluded with an analysis of set-parameter free variants of ATR_0 and related systems. This is the topic of chapter 3. It turns out that ATR_0 is not affected by the removal of set-parameters. We then focus on a set-parameter free fixed point schema (FP^-) and related systems. We pin down their proof-theoretic strengths by relating these to a variant of (ATR) among a fixed primitive recursive well ordering.

In Part II we are interested in set-theoretic analogues of questions that were treated in Part I. To this end, we introduce a range of basic set theories featuring the natural numbers as urelements and induction principles on sets and the natural numbers of various strengths. This is done in chapter 4. To interpret set-theoretic objects within second order arithmetic, we adapt the method of representation trees given in [Jäg86, Sim09]. As mentioned before, reduction principles have very natural set-theoretic counterparts. To this end, we define the schemas (Σ-Red) and (Π-Red) in chapter 6. Making use of representation trees we can then determine the effect on proof-theoretic strength when adding these reduction principles to our basic set theories. Doing so, we obtain theories with the same proof-theoretic strength as Σ^1_1-AC, ATR_0 and ATR.

In chapter 7 the set theories BETA_0 and BETA are introduced. These theories are closely related to Simpson's set-theoretic variant of ATR_0. We are then interested in the effect of adding set-theoretic reduction principles to BETA_0 and BETA. We determine the proof-theoretic strength of the resulting set theories by establishing connections to certain subsystems of second order arithmetic. The chapter is then concluded with a brief outlook on questions related to Kripke-Platek set theory.

In the Appendix A we discuss a technical question regarding the role of certain variables in the schema (ATR). It is included to provide clarity.

Part I.

Subsystems of Second Order Arithmetic

1. Preliminaries

In this chapter we establish the general setting in the context of second order arithmetic. From the syntactic side we mainly adhere to [Buc05]. Regarding the introduction of subsystems of second order arithmetic, we follow closely [Sim09]. For more technical details we refer to [TS00], and for the general background to [End01, Hin18].

1.1. The language $\mathcal{L}_2$

In this section we introduce the *language* $\mathcal{L}_2$ *of second order arithmetic* together with basic syntactic notions. $\mathcal{L}_2$ features the following *logical symbols*:

- *(propositional) connectives* $\wedge$ and $\vee$,
- the *negation symbol* $\sim$ to form negated literals,
- the *quantifier symbols* $\forall$ and $\exists$,
- the *set membership symbol* $\in$,
- the *equality symbol* $=$,
- a countable set of *number variables* $\mathrm{Var}_1 := \{\nu_i : i \in \mathbb{N}\}$,
- a countable set of *set variables* $\mathrm{Var}_2 := \{\mathcal{V}_i : i \in \mathbb{N}\}$,
- auxiliary symbols, e.g., comma and parentheses.

As *function* and *relation symbols* we use:

- *symbols* for *primitive recursive (p.r.) functions* and *relations*,
- a *unary anonymous relation symbol* Q,
- additional *relation symbols* that are introduced in certain contexts.

The *p.r. function* and *relation symbols* are inductively defined as follows, where n and k range over natural numbers (including zero):

- the n-ary *zero function symbol* 0^n is p.r.,
- the unary *successor function symbol* Succ is p.r.,
- for $1 \leq i \leq n$ the n-ary *projection function symbol* P_i^n is p.r.,
- the n-ary function symbol $\mathrm{Comp}^n(g, h_1, \dots, h_k)$ is p.r. for all k-ary p.r. function symbols g and n-ary p.r. function symbols $h_1, \dots, h_k$,
- the $(n+1)$-ary function symbol $\mathrm{Rec}^{n+1}(g, h)$ is p.r. for all n-ary p.r. function symbols g and $(n+2)$-ary p.r. function symbols h,
- the $(n+1)$-ary relation symbol $\mathrm{Rel}^{n+1}(g)$ is p.r. for all $(n+1)$-ary p.r. function symbols g.

Moreover, we will occasionally introduce additional *function symbols*, depending on the situation. The relation symbol Q serves as an auxiliary symbol, e.g., in the context of certain variants of arithmetic transfinite recursion. If no confusion arises, we add and omit parentheses freely for better readability. The *first order language* $\mathcal{L}_1$ is obtained from $\mathcal{L}_2$ by omitting the set variables of Var_2. All notions introduced refer to $\mathcal{L}_2$, but can be adapted easily to $\mathcal{L}_1$, extensions of $\mathcal{L}_2$ with additional relation symbols, and so on.

Number terms of $\mathcal{L}_2$ are introduced as usual and we use $\mathrm{Tm}(\mathcal{L}_2)$ to denote the set containing all these terms. Numerals are defined by $\overline{0} = 0^0$, $\overline{1} = \mathrm{Succ}(\overline{0})$, $\overline{2} = \mathrm{Succ}(\mathrm{Succ}(\overline{0}))$, and so on. However, if no confusion arises we simply write $0, 1, 2$, etc.. A term in which no variables occur is called *closed*. An *atomic formula* is an expression of the form $s = t$, $t \in X$ or $R(t_1, \dots, t_n)$ for any number terms $s, t, t_1, \dots, t_n$, set variable X and n-ary relation symbol R. A *literal* is either an atomic formula $\mathscr{A}$ or an expression of the form $\sim\mathscr{A}$, where $\mathscr{A}$ is an atomic formula. We use $s \neq t$ and $t \notin X$ as abbreviations for $\sim(s = t)$ and $\sim(t \in X)$, respectively. Moreover, given a binary relation symbol R, we write sRt for $R(s, t)$.

We work with formulas in negation normal form that are inductively defined as follows:

- All literals are formulas,
- if $\mathscr{A}$ and $\mathscr{B}$ are formulas, then so are $(\mathscr{A} \wedge \mathscr{B})$, $(\mathscr{A} \vee \mathscr{B})$, $\forall \nu_i \mathscr{A}$, $\exists \nu_i \mathscr{A}$, $\forall \mathcal{V}_i \mathscr{A}$ and $\exists \mathcal{V}_i \mathscr{A}$ for all $i \in \mathbb{N}$.

The following (possibly subscripted) metavariables are used:

- $h, i, j, k, l, m, n, u, v, w, x, y, z$ for number variables,
- $H, M, N, P, U, V, W, X, Y, Z$ for set variables,
- r, s, t for number terms,
- $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{F}$ for formulas.

$\mathrm{Fml}(\mathcal{L}_2)$ stands for the set of all formulas of $\mathcal{L}_2$. The negation of a formula $\mathscr{A}$, denoted $\neg\mathscr{A}$, is defined inductively by using De Morgan's laws, quantifier laws, and demanding $\neg\mathscr{B} = {\sim}\mathscr{B}$, $\neg{\sim}\mathscr{B} = \mathscr{B}$ in case $\mathscr{B}$ is an atomic formula. For convenience, we also introduce the connectives $\rightarrow$ and $\leftrightarrow$:

$$(\mathscr{A} \rightarrow \mathscr{B}) := (\neg\mathscr{A} \vee \mathscr{B}) \text{ and } (\mathscr{A} \leftrightarrow \mathscr{B}) := ((\mathscr{A} \rightarrow \mathscr{B}) \wedge (\mathscr{B} \rightarrow \mathscr{A})).$$

We determine that $\forall, \exists, \neg$ bind stronger than $\wedge, \vee$, which themselves bind stronger than $\rightarrow, \leftrightarrow$. Consider a formula $\mathscr{A}$. The set of all number and set variables with free occurrences in $\mathscr{A}$ is denoted $\mathrm{FrVar}(\mathscr{A})$. A set variable X is said to *occur positively* in $\mathscr{A}$ if no formula of the form $t \notin X$ appears as a substring of $\mathscr{A}$. In that case we also call $\mathscr{A}$ *X-positive*.

We use vector notation to denote lists of different variables, terms and other syntactic entities, e.g., $\vec{t} = t_1, \dots, t_n$, $\vec{X} = X_3, \dots, X_7$. To improve readability we use various abbreviations when writing down formulas, e.g., $\forall x_1 \forall x_2 \dots \forall x_n$ stands for $\forall \vec{x}$ and $x_1, x_2 \in Y$ for $x_1 \in Y \wedge x_2 \in Y$. Further abbreviations such as $\exists \vec{x}$, $\forall \vec{X}$, $\exists \vec{X}$ or $(\exists x \in Y)$, $(\forall x \in Y)$ are treated accordingly.

Let $\mathscr{A}$ be a formula, $s, \vec{t} = t_1, \dots, t_n$ terms, and $\vec{x} = x_1, \dots, x_n$ number variables. The *substitution of $\vec{t}$ for $\vec{x}$ in s*, written $s(\vec{x}/\vec{t})$, is defined by simultaneously replacing all occurrences of x_i by t_i for all $i = 1, \dots, n$. Similarly, the *substitution of $\vec{t}$ for free occurrences of $\vec{x}$ in $\mathscr{A}$*, denoted $\mathscr{A}(\vec{x}/\vec{t})$, is the result of simultaneously replacing all free occurrences of x_i in $\mathscr{A}$ by t_i for all $i = 1, \dots, n$.

In order to deal with substitution of set variables, we introduce the notion of class terms. A *class term* is an expression of the form $\{x : \mathscr{C}\}$ for any number variable x and formula $\mathscr{C}$. Given a list of variables $x_1, \dots, x_n$, and a list of formulas $\mathscr{C}_1, \dots, \mathscr{C}_n$, the *substitution of the class terms*

$$\{x_1 : \mathscr{C}_1\}, \dots, \{x_n : \mathscr{C}_n\}$$

for free occurrences of $X_1, \ldots, X_n$ *in* $\mathscr{A}$, denoted

$$\mathscr{A}(X_1/\mathscr{C}_1, \ldots, X_n/\mathscr{C}_n)$$

is obtained by simultaneously replacing, within $\mathscr{A}$, all occurrences of $t \in X_i$ that are not in the scope of a quantifier $\forall X_i$ or $\exists X_i$, by $\mathscr{C}_i(x_i/t)$.

Following [TS00], we identify formulas that only differ in the names of bound variables. We always assume that variable clashes in substitutions are avoided by renaming bound variables such that these variables do not occur in the terms that are to be substituted.

References to substituted variables are often dropped, e.g., we write $\mathscr{A}(\vec{t})$ instead of $\mathscr{A}(\vec{x}/\vec{t})$. The substitution of X with U in some formula $\mathscr{A}$ can be defined as $\mathscr{A}(X/\mathscr{F}(x))$ with $\mathscr{F}(x) \equiv x \in U$. In this manner, the substitution of class terms also encompasses the substitution of set variables with set variables. To simplify the notation, we just write $\mathscr{A}(X/U)$ or $\mathscr{A}(U)$ for such substitutions. If $M = \{n : \mathscr{F}(n)\}$ is the class term with defining formula $\mathscr{F}(n)$, we use $t \in M$ as abbreviation for $\mathscr{F}(t)$. If clear from context, we write $\mathscr{A}(\mathscr{F}(x))$ or $\mathscr{A}(M)$ instead of $\mathscr{A}(X/\mathscr{F}(x))$. Usually, our axioms make sure that the classes which are considered exist properly as sets. Occasionally, square instead of round brackets are used when writing down formulas, e.g., $\mathscr{F}[m, n, X]$. This indicates that only the displayed variables, m, n, X in the given example, occur freely in the respective formula. Moreover, we sometimes write $t[\vec{x}]$ to indicate that the variables occurring in the term t are among $\vec{x}$.

Next, we shall discuss the "underlying logic". We assume a classical Hilbert-style system consisting of a set of *logical axioms* and a set of *(inference) rules*. We first discuss the axioms and then turn to the rules.

The logical axioms are divided into *tautologies*, *equality axioms*, and *axioms* for *number quantifiers* and *set quantifiers*. We only consider equality on the level of number variables. For set variables X, Y, equality will be defined as:

$$X = Y :\equiv \forall n(n \in X \leftrightarrow n \in Y).$$

To define the notion of tautologies we use *propositional valuations*. Consider the truth values *true* ($\top$) and *false* ($\bot$). A propositional valuation $\mathbb{V}$ is a map from $\mathrm{Fml}(\mathcal{L}_2)$ to $\{\top, \bot\}$ satisfying the following conditions:

- $\mathbb{V}(\sim\mathscr{A}) = \begin{cases} \top & \text{if } \mathbb{V}(\mathscr{A}) = \bot, \\ \bot & \text{otherwise,} \end{cases}$ in case $\mathscr{A}$ is atomic,

- $\mathbb{V}((\mathscr{A} \wedge \mathscr{B})) = \begin{cases} \top & \text{if } \mathbb{V}(\mathscr{A}) = \mathbb{V}(\mathscr{B}) = \top, \\ \bot & \text{otherwise,} \end{cases}$
- $\mathbb{V}((\mathscr{A} \vee \mathscr{B})) = \begin{cases} \bot & \text{if } \mathbb{V}(\mathscr{A}) = \mathbb{V}(\mathscr{B}) = \bot, \\ \top & \text{otherwise.} \end{cases}$

A formula is called *tautology* if $\mathbb{V}(\mathscr{A}) = \top$ for all propositional valuations $\mathbb{V}$. Intuitively, tautologies are formulas that are true by virtue of their build-up using the propositional connectives $\wedge$, $\vee$, $\sim$.

The logical axioms are now given as follows:

Tautologies:

$$\text{(Taut)} \quad \mathscr{A}$$

for all tautologies $\mathscr{A}$.

Eequality axioms:

$$\begin{array}{ll} \text{(Refl)} & x = x, \\ \text{(Lit)} & x = y \to (\mathscr{A}(z/x) \to \mathscr{A}(z/y)), \end{array}$$

for all literals $\mathscr{A}$, and number variables x, y, z.

Axioms for number quantifiers:

$$\begin{array}{ll} (\mathrm{A}_1\forall) & \forall x \mathscr{A} \to \mathscr{A}(x/t), \\ (\mathrm{A}_1\exists) & \mathscr{A}(x/t) \to \exists x \mathscr{A}(x), \end{array}$$

for all formulas $\mathscr{A}$, number variables x, and term t, free for x in $\mathscr{A}$.

Axioms for set quantifiers:

$$\begin{array}{ll} (\mathrm{A}_2\exists) & \forall X \mathscr{A} \to \mathscr{A}(X/U), \\ (\mathrm{A}_2\forall) & \mathscr{A}(X/U) \to \exists X \mathscr{A}(X), \end{array}$$

for all formulas $\mathscr{A}$, set variables X, U, where U is free for X in $\mathscr{A}$.

By definition, a *rule* is a pair $\langle \Theta, \mathscr{A} \rangle$ such that $\Theta \cup \{\mathscr{A}\}$ is a finite set of formulas. Given a rule $\langle \Theta, \mathscr{A} \rangle$, the elements of Θ are called *premises* and $\mathscr{A}$ is called *conclusion*. Rules of the form $(\{\mathscr{A}_1, \ldots, \mathscr{A}_n\}, \mathscr{A})$ are depicted as

$$\frac{\mathscr{A}_1 \qquad \mathscr{A}_2 \qquad \cdots \qquad \mathscr{A}_n}{\mathscr{A}}$$

We use rules for *modus ponens*, *number quantifiers* and *set quantifiers*:

Rules for modus ponens:

$$\frac{\mathscr{A} \qquad \mathscr{A} \to \mathscr{B}}{\mathscr{B}}$$

for all formulas $\mathscr{A}, \mathscr{B}$.

Rules for number quantifiers:

$$(\mathrm{R}_1\forall)\ \frac{\mathscr{A} \to \mathscr{B}}{\mathscr{A} \to \forall x \mathscr{B}} \qquad (\mathrm{R}_1\exists)\ \frac{\mathscr{B} \to \mathscr{A}}{\exists x \mathscr{B} \to \mathscr{A}}$$

for all formulas $\mathscr{A}, \mathscr{B}$, and number variables $x \notin \mathrm{FrVar}(\mathscr{A})$.

Rules for set quantifiers:

$$(\mathrm{R}_2\forall)\ \frac{\mathscr{A} \to \mathscr{B}}{\mathscr{A} \to \forall X \mathscr{B}} \qquad (\mathrm{R}_2\exists)\ \frac{\mathscr{B} \to \mathscr{A}}{\exists X \mathscr{B} \to \mathscr{A}}$$

for all formulas $\mathscr{A}, \mathscr{B}$, and set variables $X \notin \mathrm{FrVar}(\mathscr{A})$.

To conclude this section we recall the notion of *derivation*. Let T be a set of formulas. A derivation from T is a finite sequence of formulas $\mathscr{D}_1, \ldots, \mathscr{D}_n$ such that for all $i = 1, \ldots, n$ one of the following conditions holds:

- $\mathscr{D}_i$ is a logical axiom,
- $\mathscr{D}_i$ is an element of T,
- $\mathscr{D}_i$ is the conclusion of a rule such that all premises of that rule occur in the sequence $\mathscr{D}_1, \ldots, \mathscr{D}_{i-1}$.

A formula $\mathscr{D}$ is called *derivable* from T if there exists a derivation from T with last formula $\mathscr{D}$. In that case we write $\mathsf{T} \vdash \mathscr{D}$ and $\mathscr{D}$ is called a *theorem of* T. Moreover, we use wordings such as "T proves $\mathscr{D}$" or "$\mathscr{D}$ is provable from T".

In general, we identify a *formal system* FS with the collection of its non-logical axioms $\mathrm{NL}(\mathsf{FS})$. Accordingly, we simply write $\mathsf{FS} \vdash \mathscr{D}$ instead of $\mathrm{NL}(\mathsf{FS}) \vdash \mathscr{D}$. Given two system $\mathsf{FS}_1, \mathsf{FS}_2$, we say that FS_1 is a subsystem of FS_2, written $\mathsf{FS}_1 \subseteq \mathsf{FS}_2$, if all theorems of FS_1 are also theorems of FS_2.

1.2. Semantics of $\mathcal{L}_2$

We briefly introduce semantic notions in order to fix the notation. For reference we refer to [End01, Hin18]. A structure $\mathcal{M}$ (in the language $\mathcal{L}_2$) consists of

- a non-empty set $|\mathcal{M}|$, called the universe of $\mathcal{M}$,
- a subset $\mathcal{S}^{\mathcal{M}}$ of the power set of $|\mathcal{M}|$, called *set universe of* $\mathcal{M}$,
- a function $f^{\mathcal{M}}\colon |\mathcal{M}|^n \to |\mathcal{M}|$ for each n-ary function symbol f,
- a relation $R^{\mathcal{M}} \subseteq |\mathcal{M}|^n$ for each n-ary relation symbol R.

Given a structure $\mathcal{M}$, an $\mathcal{M}$-assignment is a mapping

$$\mathrm{V}\colon \mathrm{Var}_1 \cup \mathrm{Var}_2 \to |\mathcal{M}| \cup \mathcal{S}^{\mathcal{M}}$$

such that $\mathrm{V}(x) \in |\mathcal{M}|$ for all number variables x, and $\mathrm{V}(X) \in \mathcal{S}^{\mathcal{M}}$ for all set variables X. The extension $\bar{\mathrm{V}}\colon \mathrm{Tm}(\mathcal{L}_2) \cup \mathrm{Var}_2 \to |\mathcal{M}| \cup \mathcal{S}^{\mathcal{M}}$ of V is given by extending V on $\mathrm{Tm}(\mathcal{L}_2)$ as usual by ensuring compatibility with functions and leaving V unchanged on Var_2. For any $\mathcal{M}$-assignment W, $a \in |\mathcal{M}|$ and $y \in \mathrm{Var}_1$, we write $\mathrm{W}(y : a)$ to denote the $\mathcal{M}$-assignment satisfying

$$\mathrm{W}(y{:}a)(x) = \begin{cases} a & \text{if } x = y, \\ \mathrm{W}(x) & \text{otherwise,} \end{cases}$$

for $x \in \mathrm{Var}_1$, and $\mathrm{W}(y : a)(X) = \mathrm{W}(X)$ for $X \in \mathrm{Var}_2$. Analogously, $\mathrm{W}(Y : A)$ denotes W with the modified value $A \in \mathcal{S}^{\mathcal{M}}$ at the argument $Y \in \mathrm{Var}_2$. All ingredients to define *"$\mathscr{A}$ is true for* V *in* $\mathcal{M}$*"* (or *"$\mathcal{M}$ satisfies $\mathscr{A}$ for* V*"*), written $\mathcal{M}, \mathrm{V} \vDash \mathscr{A}$, are now available:

- $\mathcal{M}, \mathrm{V} \vDash s = t$ if $\bar{\mathrm{V}}(s) = \bar{\mathrm{V}}(t)$,
- $\mathcal{M}, \mathrm{V} \vDash t \in X$ if $\bar{\mathrm{V}}(t) \in \bar{\mathrm{V}}(X)$,
- $\mathcal{M}, \mathrm{V} \vDash R(t_1, \dots, t_n)$ if $\langle \bar{\mathrm{V}}(t_1), \dots, \bar{V}(t_n)\rangle \in R^{\mathcal{M}}$,
- $\mathcal{M}, \mathrm{V} \vDash (\mathscr{B}_1 \wedge \mathscr{B}_2)$ if $\mathcal{M}, \mathrm{V} \vDash \mathscr{B}_1$ and $\mathcal{M}, \mathrm{V} \vDash \mathscr{B}_2$,
- $\mathcal{M}, \mathrm{V} \vDash (\mathscr{B}_1 \vee \mathscr{B}_2)$ if $\mathcal{M}, \mathrm{V} \vDash \mathscr{B}_1$ or $\mathcal{M}, \mathrm{V} \vDash \mathscr{B}_2$,
- $\mathcal{M}, \mathrm{V} \vDash \exists x \mathscr{B}$ if $\mathcal{M}, \mathrm{V}(x{:}a) \vDash \mathscr{B}$ for some $a \in |\mathcal{M}|$,

- $\mathcal{M}, \mathrm{V} \vDash \forall x \mathscr{B}$ if $\mathcal{M}, \mathrm{V}(x:a) \vDash \mathscr{B}$ for all $a \in |\mathcal{M}|$,
- $\mathcal{M}, \mathrm{V} \vDash \exists X \mathscr{B}$ if $\mathcal{M}, \mathrm{V}(X:A) \vDash \mathscr{B}$ for some $A \in \mathcal{S}^{\mathcal{M}}$,
- $\mathcal{M}, \mathrm{V} \vDash \forall X \mathscr{B}$ if $\mathcal{M}, \mathrm{V}(X:A) \vDash \mathscr{B}$ for all $A \in \mathcal{S}^{\mathcal{M}}$.

A formula $\mathscr{A}$ is called valid in $\mathcal{M}$, written $\mathcal{M} \vDash \mathscr{A}$, if $\mathcal{M}, V \vDash \mathscr{A}$ holds for all $\mathcal{M}$-assignments V. $\mathscr{A}$ is called valid if $\mathscr{A}$ is valid in all structures. $\mathscr{A}$ is *satisfiable* if there exists a structure $\mathcal{N}$ and an $\mathcal{N}$-assignment W such that $\mathcal{M}, W \vDash \mathscr{A}$. Given a formal system T, a structure $\mathcal{M}$ is called a *model of* T iff all formulas of T are valid in $\mathcal{M}$. Recall that a (formal) system is just a set of formulas. By applying Gödel's completeness theorem to the two-sorted language $\mathcal{L}_2$, the following important principle holds: A given formula $\mathscr{A}$ is a theorem of the formal system T iff $\mathscr{A}$ holds in every model of T.

1.3. The system ACA_0

With $\mathcal{L}_2$ as our underlying language, we now turn to the definition of our base theory ACA_0, which is a subsystem of the formal system Z_2 of *second order arithmetic*. Recall that we write $0, 1$ and so on for numerals representing the (standard) natural numbers. The axioms of Z_2 are as follows:

Number-theoretic axioms:

- $0^n(x_1, \ldots, x_n) = 0$,
- $\mathrm{Succ}(x) \neq 0$,
- $\mathrm{Succ}(x) = \mathrm{Succ}(y) \to x = y$,
- $\mathrm{P}^n_i(x_1, \ldots, x_n) = x_i$,
- $\mathrm{Comp}^n(g, h_1, \ldots, h_k)(x_1, \ldots, x_n)$
  $= g(h_1(x_1, \ldots, x_n), \ldots, h_k(x_1, \ldots, x_n))$,
- $\mathrm{Rec}^{n+1}(g, h)(0, x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$,

  $\mathrm{Rec}^{n+1}(g, h)(\mathrm{Succ}(m), x_1, \ldots, x_n)$
  $= h(\mathrm{Rec}^{n+1}(g, h)(m, x_1, \ldots, x_n), m, x_1, \ldots, x_n)$,
- $(\mathrm{Rel}^{n+1}(g))(x_1, \ldots, x_{n+1}) \leftrightarrow g(x_1, \ldots, x_{n+1}) \neq 0$,

with n, k ranging over natural numbers.

Induction axiom:

$$\forall X(0 \in X \land \forall n(n \in X \to \mathrm{Succ}(n) \in X) \to \forall n(n \in X)).$$

Full comprehension schema:

$$\exists X \forall n(n \in X \leftrightarrow \mathscr{A}(n))$$

for any formula $\mathscr{A}(n)$ with $X \notin \mathrm{FrVar}(\mathscr{A})$. $\mathscr{A}(n)$ may contain additional free number and set variables. The number variable n does not have to occur in $\mathscr{A}(n)$.

All formal systems in $\mathcal{L}_2$ that will be considered are subsystems of Z_2. They will only differ from Z_2 in that they lack the full comprehension axiom schema. Instead, weaker set-existence principle are introduced. It turns out that a big branch of (ordinary) mathematics can be formalised in just five subsystems of Z_2, featuring set-existence principles of increasing strength. Even more so, it can be shown that many mathematical theorems formalised this way are equivalent to one of these five set-existence principles over some base theory. Research endeavours in this direction are subsumed under the Friedman/ Simpson program of "Reverse Mathematics". The textbook by Simpson, cf. [Sim09], provides an extensive overview of these topics. For crucial contributions by Friedman we refer to [Fri75] and [Sim85].

Before defining ACA_0, we recall some basic formula types. A formula $\mathscr{A}$ is called *arithmetical* if it contains no bound set variables. $\mathscr{A}$ might contain free number and set variables. $\mathscr{A}$ is called Π^1_1 or Σ^1_1 if it is of the form $\forall X \mathscr{B}$ or $\exists X \mathscr{B}$, respectively, where $\mathscr{B}$ is arithmetical. Σ^1_k formulas are of the form

$$\exists X_1 \forall X_2 \exists \dots X_k \mathscr{B}$$

with k alternating set quantifiers and $\mathscr{B}$ being arithmetical. Π^1_k formulas are defined analogously, with the alternating quantifiers starting with $\forall$.

The *arithmetical comprehension schema* is the restriction of the full comprehension axiom schema to arithmetical formulas, i.e.,

$$\exists X \forall n(n \in X \leftrightarrow \mathscr{A}(n)), \tag{ACA}$$

where $\mathscr{A}(n)$ is arithmetical, with X not occurring freely. As before, $\mathscr{A}(n)$

might contain parameters and n might not occur at all in $\mathscr{A}(n)$.

ACA_0 is the subsystem of Z_2 resulting from restricting the full comprehension schema to the arithmetical comprehension schema. ACA is ACA_0 together with the full induction schema, i.e.,

$$(\mathscr{A}(0) \wedge \forall n(\mathscr{A}(n) \to \mathscr{A}(\mathrm{Succ}(n)))) \to \forall n \mathscr{A}(n),$$

where $\mathscr{A}(n)$ is any formula.

ACA is an acronym standing for "arithmetical comprehension axiom". Other comprehension or induction schemas will be used. We will not define these separately as the names are usually self-explanatory. Note that in ACA_0 the *arithmetical induction schema* can be deduced. Occasionally, we use PA, which is short for Peano arithmetic, to refer to the first order part of ACA_0.

1.4. Mathematical notions within ACA_0

In this section several mathematical notions are introduced. All definitions and assertions can be formalised within ACA_0. Basically, we follow [Sim09], but some adjustments are made in order to comply with [Buc05]. In particular, this concerns the coding of sequences and the introduction of *ordinal notation systems*. We reserve $+$, $\cdot$ and $<$ for p.r. symbols defining the usual addition, multiplication and "less than" relation on the natural numbers, respectively. Analogously, we define $\leq$. Within ACA_0, two sets X, Y are *equal*, written $X = Y$, if

$$\forall n(n \in X \leftrightarrow n \in Y).$$

Nat is the unique set X (up to equality) such that

$$\forall n(n \in X \leftrightarrow n = n).$$

A natural number is simply an element of Nat. As expected, Nat together with $0, 1$, addition, multiplication and $<$ forms a commutative ordered semiring with cancellation. We reserve the symbol $\mathbb{N}$ for the natural numbers in the sense of our meta theory. Note that in [Sim09], $\mathbb{N}$ is used instead of Nat, and ω instead of $\mathbb{N}$. We reserve ω to be the first limit ordinal.

A central role is occupied by finite sequences. We use a p.r. machinery to code these. More precisely, we set $\langle\rangle := 0$, and for $n > 0$ and natural numbers $u_0, \dots, u_{n-1}$,

$$\langle u_0, \dots, u_{n-1}\rangle := \mathrm{prim}(0)^{u_0+1} + \dots + \mathrm{prim}(n-1)^{u_{n-1}+1},$$

where prim is the p.r. function enumerating the prime numbers. For every $k \in \mathbb{N}$, the k-ary p.r. function mapping $u_0, \dots, u_{k-1}$ to $\langle u_0, \dots, u_{k-1}\rangle$ is p.r.. In addition, we presuppose a unary function symbol lh and the binary p.r. function symbols $*$, $(\cdot).$ and const. The function corresponding to lh gives the length of a finite sequence, $*$ concatenates finite sequences, $(\cdot).$ picks elements out of finite sequences, and const is needed for technical reasons. More precisely, we can prove in ACA_0 that:

$$\begin{gathered}\mathrm{lh}(\langle u_0, \dots, u_{n-1}\rangle) = n,\\ i < n \to ((\langle u_0, \dots, u_{n-1}\rangle)_i = u_i \wedge u_i < \langle u_0, \dots, u_{n-1}\rangle),\\ \langle u_0, \dots, u_{n-1}\rangle * \langle v_0, \dots, v_{m-1}\rangle = \langle u_0, \dots, u_{n-1}, v_0, \dots, v_{m-1}\rangle,\\ \mathrm{const}(w, \langle u_0, \dots, u_{n-1}\rangle) = \langle w, u_0, \dots, u_{n-1}\rangle.\end{gathered}$$

We use the symbol Seq to denote a unary p.r. relation containing all codes of finite sequences, i.e., for all natural numbers s,

$$s \in \mathrm{Seq} \leftrightarrow s = \langle (s)_0, \dots, (s)_{\mathrm{lh}(s)-1}\rangle.$$

As a *pairing map* we then use the p.r. function mapping u, v to $\langle u, v\rangle$. By the above remarks we can assume that the following properties hold:

- $u < \langle u, v\rangle$ and $v < \langle u, v\rangle$,
- $\langle u_1, v_1\rangle = \langle u_2, v_2\rangle \to (u_1 = v_1 \wedge u_2 = v_2)$.

A set X is called finite if $\exists k \forall n (n \in X \to n < k)$. The Cartesian product $X \times Y$ of two sets X, Y is then defined as $\{\langle u, v\rangle : u \in X \wedge v \in Y\}$. Formally, $X \times Y$ is still a subset of Nat. A function $f \colon X \to Y$ is a set $f \subseteq X \times Y$ with the usual properties, i.e.,

- $(\forall x \in X)(\exists y \in Y)(\langle x, y\rangle \in f)$,
- $\forall x, y_1, y_2 (\langle x, y_1\rangle \in f \wedge \langle x, y_2\rangle \in f \to y_1 = y_2)$.

To conclude this section we discuss how to work with orderings in our setting. A set $X \subseteq \mathrm{Nat} \times \mathrm{Nat} \subseteq \mathrm{Nat}$ is called reflexive if

$$\forall i \forall j (\langle i, j\rangle \in X \to \langle i, i\rangle \in X \land \langle j, j\rangle \in X).$$

If X is reflexive we set $\mathrm{field}(X) := \{i : \langle i, i\rangle \in X\}$ and

$$\begin{gathered} i \leq_X j \leftrightarrow \langle i, j\rangle \in X, \\ i <_X j \leftrightarrow (\langle i, j\rangle \in X \land \langle j, i\rangle \notin X). \end{gathered}$$

X is called a *countable linear ordering* if X is reflexive and for all i, j, k:

$$\begin{gathered} (i \leq_X j \land j \leq_X k) \to i \leq_X k, \\ (i \leq_X j \land j \leq_X i) \to i = j, \\ i, j \in \mathrm{field}(X) \to (i \leq_X j \lor j \leq_X i). \end{gathered}$$

X is called *well-founded* if X is reflexive and every subset of $\mathrm{field}(X)$ has a X-minimal element, i.e.,

$$(\forall M \subseteq \mathrm{field}(X))(M \neq \emptyset \to (\exists x \in M)(\forall y \in M)(y \leq_X x \to x = y)).$$

We say that X is a *countable well ordering* if it is well-founded and a countable linear ordering. We define $\mathrm{WF}(X)$, $\mathrm{LO}(X)$ and $\mathrm{WO}(X)$ to be formulas (with only X as free variable) expressing that X is, respectively, well founded, a countable linear ordering, and a countable well ordering. The term "countable " is usually left out. Observe that $\mathrm{WO}(X)$ is Π^1_1.

2. ATR_0 and relatives

2.1. The system ATR_0

The formal system ATR_0 featuring the schema (ATR) of "arithmetical transfinite recursion" is discussed next. Given a linear ordering W, we define the classes

$$(Y)_j := \{n : \langle n, j\rangle \in Y\}\,,$$
$$(Y)^{Wj} := \{\langle m, i\rangle \in Y : i <_W j\}\,.$$

Note that the defining formula for $(Y)^{Wj}$ is of the form

$$\mathscr{F}(n) \equiv \exists m, i(n = \langle m, i\rangle \wedge n \in Y \wedge i <_W j).$$

Intuitively, thinking of Y as a hierarchy along W, $(Y)_j$ contains the elements of the j-th level of Y, and $(Y)^{Wj}$ is the disjoint sum of all levels of Y up to some $j \in \mathrm{field}(W)$. By (ACA), these classes exist properly as sets. The schema (ATR) asserts the existence of a set Y that results from iterating an operator, given by an arithmetical formula $\mathscr{A}(n, j, X)$, along some well ordering W. To state this rigorously, define $\mathcal{H}_{\mathscr{A}}(W, Y)$ to be the formula expressing that $\mathrm{LO}(W)$ and

$$Y = \left\{\langle n, j\rangle : j \in \mathrm{field}(W) \wedge \mathscr{A}(n, j, (Y)^{Wj})\right\}.$$

The bound variable j will be referred to as *field variable*. Moreover, let $\mathcal{H}_{\mathscr{A}}(k, W, Y)$ be the formula stating that $\mathrm{LO}(W)$, $k \in \mathrm{field}(W)$ and

$$Y = \left\{\langle n, j\rangle : j <_W k \wedge \mathscr{A}(n, j, (Y)^{Wj})\right\}.$$

Observe that $\mathcal{H}_{\mathscr{A}}(W, Y)$ implies

$$(Y)_j = \left\{n : \mathscr{A}(n, j, (Y)^{Wj})\right\}$$

for $j \in \text{field}(W)$, and $Y_j = \emptyset$ otherwise. In particular, $Y \subseteq \text{Nat} \times \text{field}(W)$. If $\mathscr{A}(n, j, X)$ contains additional parameters other than n, X, these also occur in $\mathcal{H}_{\mathscr{A}}(W, Y)$. However, the field variable j does not occur freely in $\mathcal{H}_{\mathscr{A}}(W, Y)$. Also, since $\mathscr{A}(n, j, X)$ is arithmetical, so is $\mathcal{H}_{\mathscr{A}}(W, Y)$.

Arithmetical transfinite recursion

The system ATR_0 consists of ACA_0 plus all formulas of the form

$$\forall W(\text{WO}(W) \to \exists Y \mathcal{H}_{\mathscr{A}}(W, Y)) \tag{ATR}$$

for any arithmetical formula $\mathscr{A}(n, j, X)$. $\mathscr{A}(n, j, X)$ might contain additional free set and numbers variables, besides n, j, X. ATR is ATR_0 together with the full induction schema.

Note that we allow the field variable j to occur in $\mathscr{A}(n, j, X)$ which is iterated. It is shown in the appendix that, over ACA_0, the schema (ATR) is equivalent to a version where the field variable must not occur.

2.2. Relatives of ATR_0

In [BJ20], several axiom schemas are introduced and shown to be equivalent to (ATR) over our base theory ACA_0. In this section, these results will be restated and extended, including their proofs. When introducing an axiom schema (Sch), we write Sch_0 for $\mathsf{ACA}_0 + \text{(Sch)}$, i.e., Sch_0 denotes the formal system consisting of ACA_0 together with (Sch). Analogously, we set

$$\mathsf{Sch} := \mathsf{ACA} + \text{(Sch)}.$$

We now present all axiom schemas and put them into context. Over ACA_0, all schemas, except (Σ^1_1-Red), turn out to be equivalent to (ATR). The equivalence proofs will be given in the next section.

Fixed points for positive arithmetical clauses

The schema (FP) consists of all formulas of the form

$$\exists X \forall n(n \in Y \leftrightarrow \mathscr{A}(n, Y)), \tag{FP}$$

where $\mathscr{A}(n, X)$ is an X-positive arithmetical formula.

Interpreting $\mathscr{A}(n, X)$ as an operator mapping sets to sets, i.e.,

$$\begin{aligned} \mathrm{P}_{\mathscr{A}}\colon\ \mathbb{P}(\mathbb{N}) &\longrightarrow \mathbb{P}(\mathbb{N}) \\ X \quad &\longmapsto \{n : \mathscr{A}(n, X)\}, \end{aligned}$$

the schema (FP) asserts the existence of some fixed point of $\mathrm{P}_{\mathscr{A}}$. The corresponding system FP_0 was shown by Avigad in [Avi96] to be equivalent to ATR_0. Note that (ACA) is a special case of (FP).

Fixed points of monotone Δ^1_1 clauses

The schema (MΔ^1_1-FP) consists of all formulas of the form

$$\mathrm{Mon}_{\mathscr{A},\mathscr{B}} \to \exists Y \forall n (n \in X \leftrightarrow \mathscr{A}(n, Y)), \tag{MΔ^1_1-FP}$$

where $\mathscr{A}(n, X)$ is a Σ^1_1 formula, $\mathscr{B}(n, X)$ a Π^1_1 formula, and $\mathrm{Mon}_{\mathscr{A},\mathscr{B}}$ is defined as the formula

$$\begin{aligned} &\forall n, X (\mathscr{A}(n, X) \leftrightarrow \mathscr{B}(n, X)) \wedge \\ &\qquad\qquad \forall X, Y (X \subseteq Y \to \forall n (\mathscr{A}(n, X) \to \mathscr{A}(n, Y))). \end{aligned}$$

The schema (MΔ^1_1-FP) demands the existence of fixed points for monotone operators that are definable by a Δ^1_1 formula. Clearly, every X-positive formula $\mathscr{A}(n, X)$ induces a monotone operator since

$$\forall X, Y (X \subseteq Y \to \forall n (\mathscr{A}(n, X) \to \mathscr{A}(n, Y))),$$

hence (MΔ^1_1-FP) is a generalisation of (FP).

Weak Σ^1_1 transfinite dependent choice

The schema (w-Σ^1_1-TDC) contains exactly all formulas of the form:

$$\begin{aligned} &\forall j \forall X \exists ! Y \mathscr{A}(j, X, Y) \wedge \mathrm{WO}(W) \to \\ &\qquad \exists Z \forall j (j \in \mathrm{field}(W) \to \mathscr{A}(j, (Z)^{Wj}, (Z)_j)), \end{aligned} \tag{w-Σ^1_1-TDC}$$

with $\mathscr{A}(n, X)$ ranging over Σ^1_1 formulas.

As is common, the $\exists !$ quantifier denotes unique existence. The term "choice" is a bit artificial since the set Z can be shown to be unique by arithmetical transfinite induction, which is available in ACA_0. The name was chosen in the style of the schema (Σ^1_1-TDC), where the uniqueness condition is not requested.

Π^1_1 and Σ^1_1 reduction

For all Σ^1_1 formulas $\mathscr{A}(n)$ and all Π^1_1 formulas $\mathscr{B}(n)$, the schema (Π^1_1-Red) consists of all formulas of the form

$$\forall n(\mathscr{A}(n) \to \mathscr{B}(n)) \to \exists Y(\forall n(\mathscr{A}(n) \to n \in Y) \land \forall n(n \in Y \to \mathscr{B}(n))). \tag{Π^1_1-Red}$$

Similarly, the schema (Σ^1_1-Red) contains exactly all formulas of the form

$$\forall n(\mathscr{B}(n) \to \mathscr{A}(n)) \to \exists Y(\forall n(\mathscr{B}(n) \to n \in Y) \land \forall n(n \in Y \to \mathscr{A}(n))), \tag{Σ^1_1-Red}$$

with $\mathscr{A}(n)$ and $\mathscr{B}(n)$ as above.

In [Sim09], the equivalence of ATR_0 and $\mathsf{\Pi}^1_1$-Red_0 is shown. Simpson introduces (Π^1_1-Red) and (Σ^1_1-Red) as, respectively, Σ^1_1 and Π^1_1 separation. However, we will also study variants of these principles in set-theoretic contexts. Thus, we use a different terminology in order to better distinguish these variants from usual set-theoretic separation principles.

Δ^1_1 transfinite arithmetical recursion

For all Σ^1_1 formulas $\mathscr{A}(n, j, X)$ and all Π^1_1 formulas $\mathscr{B}(n, j, X)$, the schema (Δ^1_1-TR) is the collection of all formulas of the form

$$\forall n, j, X(\mathscr{A}(n, j, X) \leftrightarrow \mathscr{B}(n, j, X)) \to (\mathrm{WO}(W) \to \exists Y \mathcal{H}_{\mathscr{A}}(W, Y)) \tag{Δ^1_1-TR}$$

Analogously to ($\mathrm{M}\Delta^1_1$-FP) being a generalisation of (FP), the above definition is a strengthening of (ATR) in the sense that the recursion takes place over Δ^1_1 formulas.

2.3. Equivalence proofs

In this section we will show that all axiom schemas introduced in the previous section are equivalent over ACA_0. These results are partially presented in [BJ20]. In that context, it is convenient to introduce the notion of Σ^1 and Π^1 formulas.

Definition 2.1. The class of Σ^1 formulas is the smallest class of $\mathcal{L}_2$ formulas that contains the arithmetical formulas, and is closed under the connectives $\wedge$ and $\vee$, existential and universal number quantification, as well as existential set quantification. The class of Π^1 formulas is defined exactly as the Σ^1 formulas, but instead of closure under existential set quantification, we demand closure under universal set quantification.

We continue by collecting well established results that will play a role later. These involve some additional axiom schemas presented below.

Comprehension for Δ^1_1 formulas: The schema (Δ^1_1-CA) comprises all formulas of the form

$$\forall n(\mathscr{A}(n) \leftrightarrow \mathscr{B}(n)) \to \exists X \forall n(n \in X \leftrightarrow \mathscr{A}(n)), \qquad (\Delta^1_1\text{-CA})$$

where $\mathscr{A}(n)$ is a Σ^1_1 formula and $\mathscr{B}(n)$ a Π^1_1 formula.
Σ^1_1 *choice:* The schema (Σ^1_1-AC) comprises all formulas

$$\forall n \exists Y \mathscr{A}(n, Y) \to \exists Y \forall n \mathscr{A}(n, (Y)_n), \qquad (\Sigma^1_1\text{-AC})$$

with $\mathscr{A}(n)$ ranging over Σ^1_1 formulas.

Theorem 2.2. *The following assertions hold.*

(a) (ATR), (FP) *and* (Π^1_1-Red) *are equivalent over* $\mathsf{ACA_0}$.

(b) $\mathsf{ATR_0}$ *proves the schemas* (Δ^1_1-CA) *and* (Σ^1_1-AC).

(c) $\mathsf{\Sigma^1_1}$-AC *proves all instances of* (Σ^1_1-Red).

(d) For any Σ^1_1 formula $\mathscr{A}(X)$, $\mathsf{ACA_0}$ *proves*

$$\neg \forall X(\mathscr{A}(X) \leftrightarrow \mathrm{WO}(X)).$$

For reference we refer to [Avi96, Sim09]. We continue with the following observation.

Lemma 2.3. $\mathsf{ATR_0} \subseteq \mathsf{w}\text{-}\mathsf{\Sigma^1_1}\text{-}\mathsf{TDC_0}$.

Proof. Working in $\mathsf{w}\text{-}\mathsf{\Sigma^1_1}\text{-}\mathsf{TDC_0}$, let $\mathscr{A}(n, j, X)$ be an arithmetical formula. We define the arithmetical formula

$$\mathscr{B}(j, X, Y) :\equiv Y = \{n : \mathscr{A}(n, j, X)\}\,.$$

Note that by (ACA), $\forall j \forall X \exists! Y \mathscr{B}(j, X, Y)$. Next, consider a well ordering W. Applying (w-Σ^1_1-TDC) yields a set Z such that

$$\mathscr{B}(j, (Z)^{Wj}, (Z)_j)$$

for all $j \in \mathrm{field}(W)$. By definition of $\mathscr{B}(j, X, Y)$ we have

$$(Z)_j = \left\{n : \mathscr{A}(n, j, (Z)^{Wj})\right\}. \tag{2.1}$$

Using (ACA), the set $(Z)^W := \{\langle n, j\rangle \in Z : j \in \mathrm{field}(W)\}$ exists and satisfies $\mathcal{H}_{\mathscr{A}}(W, (Z)^W)$ by (2.1). This establishes (ATR). □

The following corollary will facilitate working with Σ^1 and Π^1 formulas on certain occasions. Details can be found in [Sim09]. By Theorem 2.2(b) and Lemma 2.3 we have $\mathsf{w}\text{-}\mathsf{\Sigma}^1_1\text{-}\mathsf{TDC}_0 \vdash (\Sigma^1_1\text{-AC})$.

Corollary 2.4. *The following assertions are provable in $\mathsf{\Sigma}^1_1\text{-}\mathsf{AC}_0$, and therefore in particular also in $\mathsf{w}\text{-}\mathsf{\Sigma}^1_1\text{-}\mathsf{TDC}_0$, $\mathsf{\Delta}^1_1\text{-}\mathsf{TR}_0$ and $\mathsf{\Pi}^1_1\text{-}\mathsf{Red}$.*

(a) Every Σ^1 formula is provably equivalent to a Σ^1_1 formula, and every Π^1 formula to a Π^1_1 formula. In both cases, the free variables remain unchanged.

(b) The schema (Δ^1_1-CA).

The next lemma is rather technical. It enables us to embed $\mathsf{M\Delta}^1_1\text{-}\mathsf{FP}_0$ into $\mathsf{w}\text{-}\mathsf{\Sigma}^1_1\text{-}\mathsf{TDC}_0$.

Lemma 2.5. *Working in $\mathsf{w}\text{-}\mathsf{\Sigma}^1_1\text{-}\mathsf{TDC}_0$, consider a Σ^1_1 formula $\mathscr{A}(n, X)$ and a Π^1_1 formula $\mathscr{B}(n, X)$ satisfying*

(a) $\forall n \forall X(\mathscr{A}(n, X) \leftrightarrow \mathscr{B}(n, X))$,

(b) $\forall X, Y(X \subseteq Y \to \forall n(\mathscr{A}(n, X) \to \mathscr{A}(n, Y)))$.

Let $\mathscr{C}(W, X)$ be the formula stating that for all $j \in \mathrm{field}(W)$ the following hold:

(C1) $\mathrm{LO}(X)$,

(C2) $(X)_j = \{n : \mathscr{A}(n, \bigcup\{(X)_i : i <_W j\})\}$,

(C3) $\forall i(i <_W j \to (X)_i \subseteq (X)_j)$,

(C4) $\forall n(n \in (X)_j \to (\exists k \leq_W j)(n \in (X)_k \wedge n \notin \bigcup\{(X)_i : i <_W k\}))$.

Then $\mathsf{w}\text{-}\mathsf{\Sigma}^1_1\text{-}\mathsf{TDC}_0$ *proves for any set* W*,*

$$\mathrm{WO}(W) \to \exists Y \mathscr{C}(W, Y).$$

Proof. Suppose we work in $\mathsf{w}\text{-}\mathsf{\Sigma}^1_1\text{-}\mathsf{TDC}_0$. Let W be a well ordering. Aiming for an application of (w-Σ^1_1-TDC), we define the formula

$$\mathscr{D}(X, Y) :\equiv \exists U(U = \{n : \exists i(\langle n, i\rangle \in X)\} \wedge Y = \{n : \mathscr{A}(n, U)\}).$$

By Corollary 2.4 and (a), (w-Σ^1_1-TDC) is applicable to $\mathscr{D}(X, Y)$: First, note that $\mathscr{D}(X, Y)$ is equivalent to a Σ^1 formula and therefore also to a Σ^1_1 formula. Moreover, by (Δ^1_1-CA) and (a), the classes

$$\{n : \exists i(\langle n, i\rangle \in X)\} \quad \text{and} \quad \{n : \mathscr{A}(n, U)\}$$

exist properly as sets. It follows that $\forall X \exists! Y \mathscr{D}(X, Y)$. Thus, there exists a set Z satisfying

$$\mathscr{D}((Z)^{Wj}, (Z)_j) \tag{2.2}$$

for any $j \in \mathrm{field}(W)$. We will show that $\mathscr{C}(W, Z)$. Trivially, (C1) holds since $\mathrm{WO}(W)$. Unwrapping (2.2) gives

$$\exists U(U = \left\{n : \exists i(\langle n, i\rangle \in (Z)^{Wj})\right\} \wedge (Z)_j = \{n : \mathscr{A}(n, U)\}),$$

which amounts to

$$(Z)_j = \left\{n : \mathscr{A}\left(n, \bigcup\{(Z)_i : i <_W j\}\right)\right\},$$

i.e., (C2) holds. Next, (C3) follows immediately by (C2), the assumption (b), and the trivial observation that for $i <_W j$,

$$\bigcup\{(Z)_k : k <_W i\} \subseteq \bigcup\{(Z)_k : k <_W j\}.$$

Finally, (C4) is a direct consequence of $\mathrm{WO}(W)$: First, observe that the set

$$M := \{j \in \mathrm{field}(W) : n \in (Z)_j\}$$

exists by (ACA) for any number n. Moreover, if $M \neq \emptyset$, setting k to be the W-minimal element of M, it follows directly that

$$n \in (Z)_k \wedge n \notin \bigcup\{(Z)_i : i <_W k\}.$$

This finishes the proof. $\square$

Theorem 2.6. $\mathsf{M}\Delta^1_1\text{-}\mathsf{FP}_0 \subseteq \mathsf{w}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0$.

Proof. Working in $\mathsf{w}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0$, let $\mathscr{A}(n, X)$ be a Σ^1_1 formula, and $\mathscr{B}(n, X)$ a Π^1_1 formula satisfying

(a) $\forall n \forall X(\mathscr{A}(n, X) \leftrightarrow \mathscr{B}(n, X))$,

(b) $\forall X, Y(X \subseteq Y \to \forall n(\mathscr{A}(n, X) \to \mathscr{A}(n, Y)))$.

Let $\mathscr{C}(W, X)$ be as defined in Lemma 2.5 such that $\mathsf{w}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0$ proves

$$\mathrm{WO}(W) \to \exists Y \mathscr{C}(W, Y).$$

By Corollary 2.4, the formula $\exists Y \mathscr{C}(W, Y)$ is equivalent to a Σ^1_1 formula, therefore by Theorem 2.2(d), there exists sets R and Z such that

$$\neg\,\mathrm{WO}(R) \wedge \mathscr{C}(R, Z).$$

In particular, $\mathrm{LO}(R)$. This will be used throughout in the following. Since $\neg\,\mathrm{WO}(R)$, there exists $M \subseteq \mathrm{field}(R)$ such that $M \neq \emptyset$ and $(\forall j \in M)(\exists i \in M)(i <_R j)$. Using M as parameter, we define

$$U := \{i : (\exists j \in M)(j \leq_R i)\}\,.$$

Clearly, $U \neq \emptyset$ and U is upwards closed. Furthermore, we set

$$V := \{j : (\forall i \in U)(j <_R i)\}\,.$$

U and V exist by (ACA). Intuitively, V consists of everything below U. Obiously, V is downwards closed, U and V are disjoint, and $U \cup V = \mathrm{field}(R)$. Using (ACA) once more, we consider the sets

$$T_0 := \bigcap\{(Z)_i : i \in U\} \quad \text{and} \quad T_1 := \bigcup\{(Z)_i : i \in V\}.$$

We claim that $T_0 = T_1$. $T_1 \subseteq T_0$ follows immediately by definition and (C3). Conversely, assume $n \in T_0$ and let $j \in U \neq \emptyset$. In particular, we then have $n \in (Z)_j$. By (C4), there exists $k \leq_R j$ such that

$$n \in (Z)_k \wedge n \notin \bigcup\{(Z)_i : i <_W k\}.$$

We claim that $k \in V$. To see this suppose $i <_R k$. The above implies $n \notin (Z)_i$, hence $i \notin U$ since $n \in T_0$. It follows that

$$(\forall i <_R k)(i \in V).$$

This implies that $k \in V$ since otherwise, k would be the R-minimal element of U. Thus, we have $n \in T_1$, which shows that $T_0 \subseteq T_1$. Finally, we will prove that

(c) $\forall n(\mathscr{A}(n, T_0) \to n \in T_0)$,

(d) $\forall n(n \in T_1 \to \mathscr{A}(n, T_1))$.

For (c) we assume $\mathscr{A}(n, T_0)$ and consider $j \in U$. Since U has no R-minimal element, there exists $i \in U$ with $i <_R j$. By (b) and definition of T_0 we then get $\mathscr{A}(n, (Z)_i)$. Using (b) once again gives $\mathscr{A}(n, \bigcup\{(Z)_i : i <_R j\})$, which by (C2) yields $n \in (Z)_j$. As $j \in U$ was arbitrary, we can conclude that $n \in T_0$. For (d), suppose $n \in T_1$, i.e., $j \in (Z)_j$ for some $j \in V$. By (C2) we have

$$\mathscr{A}\left(n, \bigcup\{(Z)_i : i <_R j\}\right).$$

V is downwards closed, hence $\bigcup\{(Z)_i : i <_R j\} \subseteq T_1$. Applying (b) yields $\mathscr{A}(n, T_1)$. Thus, we showed (d). Since $T_0 = T_1$, it follows by (c) and (d) that T_0 is a fixed point for $\mathscr{A}(n, X)$. This establishes the schema (MΔ^1_1-FP). □

Theorem 2.7. $\mathsf{w}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0 \subseteq \Delta^1_1\text{-}\mathsf{TR}_0$.

Proof. We work in $\Delta^1_1\text{-}\mathsf{TR}_0$ and assume $\mathrm{WO}(W)$. Moreover,

$$\forall j \forall X \exists! Y \mathscr{A}(j, X, Y), \tag{2.3}$$

where $\mathscr{A}(j, X, Y)$ is some Σ^1_1 formula. Define the formulas

$$\mathscr{B}(n, j, X) :\equiv \exists Y(\mathscr{A}(j, X, Y) \wedge n \in Y),$$
$$\mathscr{C}(n, j, X) :\equiv \forall Y(\mathscr{A}(j, X, Y) \to n \in Y),$$

where n does not occur in $\mathscr{A}(j, X, Y)$. By (2.3) we can infer that

$$\forall n, j \forall X(\mathscr{B}(n, j, X) \leftrightarrow \mathscr{C}(n, j, X)).$$

Note that $\mathscr{B}(n, j, X)$ is Σ^1, and $\mathscr{C}(n, j, X)$ is Π^1. Thus, by Corollary 2.4 we can apply (Δ^1_1-TR). So there exists a set Z such that for any $j \in \text{field}(W)$

$$\begin{aligned}(Z)_j &= \left\{n : \mathscr{B}(n, j, (Z)^{Wj})\right\}\\ &= \left\{n : \exists Y(\mathscr{A}(j, (Z)^{Wj}, Y) \wedge n \in Y)\right\}.\end{aligned}$$

By (2.3) we then have $\mathscr{A}(j, (Z)^{Wj}, (Z)_j)$, which validates (w-$\Sigma^1_1$-TDC). $\square$

We continue with deriving $\mathsf{\Delta^1_1\text{-}TR_0}$ in $\mathsf{\Pi^1_1\text{-}Red_0}$. The following two lemmas will be useful in achieving that.

Lemma 2.8. *In $\mathsf{\Delta^1_1\text{-}CA_0}$, transfinite induction for Δ^1_1 formulas is provable, i.e., for any Σ^1_1 formula $\mathscr{A}(n)$, and any Π^1_1 formula $\mathscr{B}(n)$ such that $\forall n(\mathscr{A}(n) \leftrightarrow \mathscr{B}(n))$, we can prove that*

$$(\mathrm{WO}(W) \wedge \forall j((\forall i <_W j)\mathscr{A}(i) \to \mathscr{A}(j))) \to \forall n \mathscr{A}(n).$$

Proof. In $\mathsf{\Delta^1_1\text{-}CA_0}$, let the formulas $\mathscr{A}(n)$ and $\mathscr{B}(n)$ be as stated. Assume $\mathrm{WO}(W)$ and $\forall j((\forall i <_W j)\mathscr{A}(i) \to \mathscr{A}(j))$. By ($\Delta^1_1$-CA), consider the set $M := \{n : \neg\mathscr{A}(n)\}$. Suppose $M \neq \emptyset$. Then also $N := M \cap \text{field}(W) \neq \emptyset$. N clearly exists by (ACA). By $\mathrm{WO}(W)$, let $k \in N$ be W-minimal in N. Thus, we have $\neg\mathscr{A}(k)$ and $(\forall i <_W k)\mathscr{A}(i)$. By assumption the latter implies $\mathscr{A}(k)$, a contradiction, hence $M = \emptyset$, i.e., $\forall n \mathscr{A}(n)$. $\square$

Lemma 2.9. *The following is provable in $\mathsf{\Delta^1_1\text{-}CA_0}$, and hence also in $\mathsf{\Pi^1_1\text{-}Red_0}$: Given a Σ^1_1 formula $\mathscr{A}(n, j, X)$ and a Π^1_1 formula $\mathscr{B}(n, j, X)$ such that*

$$\forall n, j \forall X(\mathscr{A}(n, j, X) \leftrightarrow \mathscr{B}(n, j, X)),$$

it follows that

$$(\forall k \in \text{field}(W))\mathcal{H}_{\mathscr{A}}(k, W, (Y)^{Wk}) \to \exists Z \mathcal{H}_{\mathscr{A}}(W, Z).$$

Proof. We work in $\mathsf{\Delta^1_1\text{-}CA_0}$. Consider a set Y such that

$$(\forall k \in \text{field}(W))\mathcal{H}_{\mathscr{A}}(k, W, (Y)^{Wk}). \tag{2.4}$$

In the following, we fix some $j \in \text{field}(W)$. Note that for $i <_W j$,

$$(Y)_i = ((Y)^{Wj})_i \quad \text{and} \quad (Y)^{Wi} = ((Y)^{Wj})^{Wi}.$$

By (2.4) we have in particular $\mathcal{H}_{\mathscr{A}}(j, W, (Y)^{Wj})$, hence

$$(Y)_i = \left\{n : \mathscr{A}(n, j, (Y)^{Wi})\right\}.$$

If W has no maximal element, the above argument immediately implies that $\mathcal{H}_{\mathscr{A}}(W, Y \cap (\text{Nat} \times \text{field}(W)))$. Otherwise, let l be the maximum of W. By Corollary 2.4, using (Δ^1_1-CA) the following set exists properly:

$$Z := Y \cup \left\{\langle n, l\rangle : \mathscr{A}(n, l, (Y)^{Wl})\right\}.$$

Note that $(Z)^{Wl} = (Y)^{Wl}$. The definition of Z immediately yields

$$(Z)_l = \left\{n : \mathscr{A}(n, l, (Z)^{Wl})\right\}.$$

As above, we also have $(Z)_i = \left\{n : \mathscr{A}(n, i, (Z)^{Wi})\right\}$ for all $i <_W l$. Altogether, it follows that $\mathcal{H}_{\mathscr{A}}(W, Z \cap (\text{Nat} \times \text{field}(W)))$, finishing the proof. □

Theorem 2.10. $\Delta^1_1\text{-}\mathsf{TR}_0 \subseteq \Pi^1_1\text{-}\mathsf{Red}_0$.

Proof. Working in Π^1_1-Red, let $\mathscr{A}(n, j, X)$ be a Σ^1_1 formula and $\mathscr{B}(n, j, X)$ a Π^1_1 formula satisfying

$$\forall n, j \forall X(\mathscr{A}(n, j, X) \leftrightarrow \mathscr{B}(n, j, X)). \tag{2.5}$$

Moreover, we consider a well ordering W. Now, we define

$$\begin{aligned}
\mathscr{D}_0(m) :\equiv \exists n, j \exists Y(m = \langle n, j\rangle \wedge j \in \text{field}(W) \wedge \\
\mathcal{H}_{\mathscr{A}}(j, W, (Y)^{Wj}) \wedge \mathscr{A}(n, j, (Y)^{Wj})), \\
\mathscr{D}_1(m) :\equiv \forall n, j \forall Y(m = \langle n, j\rangle \wedge j \in \text{field}(W) \wedge \\
\mathcal{H}_{\mathscr{A}}(j, W, (Y)^{Wj}) \to \mathscr{A}(n, j, (Y)^{Wj})).
\end{aligned}$$

By Corollary 2.4, $\mathscr{D}_0(m)$ is provably equivalent to a Σ^1_1, and $\mathscr{D}_1(m)$ to a Π^1_1 formula by (2.5). Applying (Π^1_1-Red), we obtain a set Y such that

(a) $\forall m(\mathscr{D}_0(m) \to m \in Y)$,

(b) $\forall m(m \in Y \to \mathscr{D}_1(m))$.

We will show that the following holds for all k:

$$k \in \text{field}(W) \to \mathcal{H}_{\mathscr{A}}(k, W, (Y)^{Wk}).$$

By Corollary 2.4 and (2.5), the above formula is equivalent to a Σ^1_1 and a Π^1_1 formula. We can therefore use transfinite induction along W, cf. Lemma 2.8. In view of Lemma 2.9, we then get a proper witness for the given instance of (Δ^1_1-TR). To carry out the induction, let $k \in \text{field}(W)$ and assume that

$$(\forall j <_W k)\mathcal{H}_{\mathscr{A}}(j, W, (Y)^{Wj}).$$

Let $V := \{\langle i, j\rangle \in W : j <_W k\}$. Since W is a well ordering, so is V and the above is equivalent to

$$(\forall j \in \text{field}(V))\mathcal{H}_{\mathscr{A}}(j, V, (Y)^{Vj}).$$

Applying Lemma 2.9, we obtain a set Z satisfying $\mathcal{H}_{\mathscr{A}}(V, Z)$. This amounts to $\mathcal{H}_{\mathscr{A}}(k, W, Z)$. It remains to show that $Z = (Y)^{Wk}$. Consider $j <_W k$. It suffices to check that $(Z)_j = (Y)_j$. Suppose $n \in (Z)_j$, i.e., $\mathscr{A}(n, j, (Z)^{Wj})$. Observe that $\mathcal{H}_{\mathscr{A}}(j, W, (Z)^{Wj})$, hence we have $\mathscr{D}_0(\langle n, j\rangle)$, which by (a) implies $n \in (Y)_j$. Conversely, suppose $n \in (Y)_j$. By (b) we have $\mathscr{D}_1(\langle n, j\rangle)$. Since $\mathcal{H}_{\mathscr{A}}(j, W, (Z)^{Wj})$ we get $\mathscr{A}(n, j, (Z)^{Wj})$, i.e., $n \in (Z)_j$. This concludes the induction, and therefore the proof by Lemma 2.9. □

Finally, as mentioned in the introduction of $\mathsf{M}\Delta^1_1\text{-}\mathsf{FP}_0$, we can prove the following.

Proposition 2.11. $\mathsf{FP}_0 \subseteq \mathsf{M}\Delta^1_1\text{-}\mathsf{FP}_0$.

Proof. Working in $\mathsf{M}\Delta^1_1\text{-}\mathsf{FP}_0$, let $\mathscr{A}(n, X)$ be an X-positive arithmetical formula. By induction on the build-up of $\mathscr{A}(n, X)$, it follows that

$$\forall X, Y(X \subseteq Y \to \forall n(\mathscr{A}(n, X) \to \mathscr{A}(n, Y))),$$

hence (MΔ^1_1-FP) is applicable to $\mathscr{A}(n, X)$. This yields a fixed point for $\mathscr{A}(n, X)$, establishing (FP). □

In view of Theorem 2.2(a), Theorem 2.6, Theorem 2.7, Theorem 2.10 and Proposition 2.11 we get the following result.

Corollary 2.12. *Over* ACA_0, *the schemas*

$$(\mathrm{ATR}), (\mathrm{FP}), (\mathrm{M}\Delta^1_1\text{-FP}), (\text{w-}\Sigma^1_1\text{-TDC}), (\Delta^1_1\text{-TR}) \textit{ and } (\Pi^1_1\text{-Red})$$

are equivalent.

3. ATR_0 without set-parameters

In the previous section we showed that certain formal systems featuring different set-theoretic principles are all equivalent to ATR_0. We now turn to formal systems related to set-parameter free relatives of ATR_0. In contrast to before, we will not prove the equivalence of these systems, but rather characterise these up to proof-theoretic strength. We start with introducing the required concepts and defining the relevant systems.

3.1. The systems ATR_0^- and ATR^-

In this section set-parameter free variants of ATR_0 and related systems will be discussed. The central schema and corresponding formal systems are defined next.

Arithmetical transfinite recursion without set parameters

The axiom schema (ATR^-) consists of all formulas

$$\forall W(\mathrm{WO}(W) \to \exists Y \mathcal{H}_{\mathscr{A}}(W, Y)), \tag{ATR$^-$}$$

where $\mathscr{A}(n, j, X)$ is arithmetical with no free set variable occurring freely, except X. The corresponding systems extending ACA_0 is denoted ATR_0^-. ATR^- is ATR_0^- together with induction for all $\mathcal{L}_2$ formulas.

It turns out that ATR_0^- is as strong as ATR_0. This will be shown in the next theorem. The proof idea is that when recursively iterating an arithmetical formula $\mathscr{A}$ with set parameters $\vec{U}$ along some well ordering W, the parameters $\vec{U}$ can be coded into a new well ordering V. Without loss of generality, we can assume that there is just one parameter U. Intuitively, V consists of U ordered by $\leq$, followed by field(W), ordered by $\leq_W$. The iteration is then taken along V over a modified formula $\mathscr{A}^*$ without set parameters. $\mathscr{A}^*$ is defined such that when iterated along V, U is getting copied element by element and can therefore be retrieved. When the iteration arrives at field(W), the original formula $\mathscr{A}$ is being iterated, where U

is retrieved by referring to the respective lower part of the hierarchy. The proper statement and the formal proof are presented next.

Theorem 3.1. *The systems* ATR_0 *and* ATR_0^- *are equivalent.*

Proof. Clearly, $\mathsf{ATR}_0^- \subseteq \mathsf{ATR}_0$. For the converse direction, we work in ATR_0^-. Let W be a well ordering and

$$\mathscr{A}(n, j, X) \equiv \mathscr{A}(n, j, X, U_0, \dots, U_l)$$

arithmetical, with only the indicated set variables occurring freely. The main idea will be to code W and the parameters $U_0, \dots, U_l$ into a new well ordering V. Without loss of generality, we can assume that there is only one set parameter U besides X. Otherwise, we consider the disjoint union

$$U := \left\{ \langle n, \bar{0} \rangle : n \in U_0 \right\} \cup \ldots \cup \left\{ \langle n, \bar{l} \rangle : n \in U_l \right\},$$

where $\bar{0}, \dots, \bar{l}$ are numerals, cf. section 1.1. Obviously, by (ACA), U exists, and conversely, U_i can be retrieved from U for $i = 0, \dots, l$.

Our goal is to derive a set Z satisfying $\mathcal{H}_{\mathscr{A}}(W, Z)$, where $\mathscr{A}(n, j, X, U)$ is arithmetical, with no other free set variables besides X, U. We let

$$\begin{aligned} V := {} & \{ \langle \langle m, 0 \rangle, \langle n, 0 \rangle \rangle : m, n \in U \wedge m \leq n \} \\ & \cup \{ \langle \langle m, 0 \rangle, \langle n, 1 \rangle \rangle : m \in U \wedge n \in \mathrm{field}(W) \} \\ & \cup \{ \langle \langle m, 1 \rangle, \langle n, 1 \rangle \rangle : m, n \in \mathrm{field}(W) \wedge m \leq_W n \}, \end{aligned}$$

where, according to our convention, 0 stands for $\bar{0}$ and 1 for $\bar{1}$. Clearly, V exists by (ACA) and is a well ordering since W is one and every set is well ordered by $\leq$; a consequence of the induction axiom. The field of V consists of the disjoint union of U and field(W). Next, we define the arithmetical formula

$$\begin{aligned} \mathscr{A}^*(n, j, X) :\equiv {} & j = \langle n, 0 \rangle \vee \\ & \exists v (j = \langle v, 1 \rangle \wedge \mathscr{A}(n, v, [X]_f, [X]_p)), \end{aligned}$$

where $[X]_f$ and $[X]_p$ are defined as

$$\begin{aligned} [X]_f &:= \{ \langle n, v \rangle : \langle n, \langle v, 1 \rangle \rangle \in X \}, \\ [X]_p &:= \{ n : \langle n, \langle n, 0 \rangle \rangle \in X \}. \end{aligned}$$

Clearly, $[X]_f$ and $[X]_p$ exist as sets by (ACA). The subscripts f and p stand for, respectively, "field" and "parameter". Let Y be the result of applying (ATR$^-$) to V and $\mathscr{A}^*(n, j, X)$, hence $Y \subseteq \mathbb{N} \times \text{field}(V)$ and for all $(w, k) \in \text{field}(V)$,

$$(Y)_{\langle w,k\rangle} = \left\{n : \mathscr{A}^*(n, \langle w, k\rangle, (Y)^{V\langle w,k\rangle})\right\}.$$

By definition of V, k ranges over $0, 1$. For $k = 0$ the above unfolds to

$$(Y)_{\langle w,0\rangle} = \{n : \langle w, 0\rangle = \langle n, 0\rangle\} = \{w\},$$

with w ranging over U. It follows that

$$[Y]_p = \{n : \langle n, \langle n, 0\rangle\rangle \in Y\} = \left\{n : n \in (Y)_{\langle n,0\rangle}\right\} = U.$$

If $k = 1$, by definition of $\mathscr{A}^*(n, j, X)$, we have $n \in (Y)_{\langle w,1\rangle}$ iff

$$\mathscr{A}(n, \langle w, 1\rangle, [(Y)^{V\langle w,1\rangle}]_f, [(Y)^{V\langle w,1\rangle}]_p).$$

By properties of Y and V we get

$$[(Y)^{V\langle w,1\rangle}]_p = [Y]_p = U.$$

One can easily verify that $[(Y)^{V\langle w,1\rangle}]_f = ([Y]_f)^{Ww}$ as

$$\begin{aligned}
[(Y)^{V\langle w,1\rangle}]_f &= \left\{\langle n, v\rangle : \langle n, \langle v, 1\rangle\rangle \in (Y)^{V\langle w,1\rangle}\right\} \\
&= \{\langle n, v\rangle : \langle n, \langle v, 1\rangle\rangle \in Y \wedge \langle v, 1\rangle <_V \langle w, 1\rangle\} \\
&= \{\langle n, v\rangle : \langle n, \langle v, 1\rangle\rangle \in Y \wedge v <_W w\} \\
&= \left\{\langle n, v\rangle : \langle n, v\rangle \in [Y]_f \wedge v <_W w\right\} \\
&= ([Y]_f)^{Ww}.
\end{aligned}$$

Moreover, $([Y]_f)_w = (Y)_{\langle w,1\rangle}$. Altogether, it follows that

$$([Y]_f)_w = \left\{n : \mathscr{A}^*(n, \langle w, l\rangle, ([Y]_f)^{Ww}, U)\right\}$$

with w ranging over field(W). Clearly, $[Y]_f \subseteq \mathbb{N} \times \text{field}(W)$. Together with the above, this establishes that $[Y]_f$ is the desired set, finishing the proof. $\square$

The following is an immediate consequence of Theorem 2.2(a) and Theorem 3.1:

Corollary 3.2. *The following equivalences hold:*

(a) The systems ATR_0^-, ATR_0 *and* FP_0 *are equivalent.*

(b) The systems ATR^-, ATR *and* FP *are equivalent.*

3.2. Proof-theoretic strength of a theory

In this section, the notion of proof-theoretic ordinal of a formal system will be introduced. We use $\varphi_\cdot(\cdot)$ to denote the *Veblen functions*. Recall that $\varepsilon_0 := \varphi_1(0)$ is the first ordinal such that $\omega^{\varepsilon_0} = \varepsilon_0$, where ω denotes the first limit ordinal. Γ_0 is the *first strongly critical ordinal*, i.e., Γ_0 is the least α such that $\varphi_\alpha(0) = \alpha$. The *order type* of a well ordering W is the unique ordinal α such that there exists an order-preserving bijection between W and α. Before we can define the proof-theoretic strength of a theory, we need some additional notions.

Definition 3.3. Let $\prec$ be a binary p.r. relation. We define the following $\mathcal{L}_2$ formulas:

$$\begin{aligned}\mathrm{Prog}(\prec, X) &:\equiv \forall w((\forall v \prec w)(v \in X) \to w \in X),\\ \mathrm{TI}(\prec, X) &:\equiv \mathrm{Prog}(\prec, X) \to \forall w(w \in X).\end{aligned}$$

We now have the ingredients to define the proof-theoretic ordinal of a formal system.

Definition 3.4. Let T be a formal system in a language $\mathcal{L}$ that contains $\mathcal{L}_1$. In particular, $\mathcal{L}$ features the p.r. function and relation symbols, as well as the anonymous unary relation variable Q.

(a) An ordinal α is called *provable within* T, if there exists a (strict) p.r. well ordering $\prec$ of order type α, such that

$$\mathsf{T} \vdash \mathrm{TI}(\prec, Q).$$

(b) The *proof-theoretic ordinal of* T, denoted $|\mathsf{T}|$, is the least ordinal which is not provable within T.

To actually determine the proof-theoretic strength of a given system, we will work with the (strict) p.r. well ordering $\prec$, as introduced in [Buc05]. Henceforth, the meaning of $\prec$ is therefore fixed. The well ordering $\prec$ is large enough for our purposes. The symbol $\preceq$ denotes the non-strict version of $\prec$. We continue with introducing essential notions regarding $\prec$. All definitions are made w.r.t. some appropriate formal system T, cf. Definition 3.4. An *ordinal term* is a closed number term t such that $t \in \mathrm{field}(\preceq)$. Any ordinal term corresponds to an ordinal in the sense of our meta theory. In the following, Greek letters $\alpha, \beta, \gamma, \mu, \xi$ are used for number variables. For convenience, we introduce the additional notions,

$$\begin{aligned}
\alpha \subseteq X &:\equiv (\forall \xi \prec \alpha)(\xi \in X), \\
\mathrm{Prog}(X) &:\equiv \forall \alpha(\alpha \subseteq X \rightarrow \alpha \in X), \\
\mathrm{TI}(\alpha, X) &:\equiv \mathrm{Prog}(X) \rightarrow \alpha \subseteq X, \\
\mathrm{WO}(\alpha) &:\equiv \forall X \, \mathrm{TI}(\alpha, X).
\end{aligned}$$

Note that in $\mathsf{ACA_0}$, $\mathrm{WO}(\alpha)$ iff the restriction of $\prec$ (or rather $\preceq$) to all elements below α is a well ordering, as defined in section 1.4. If t is an ordinal term such that $\mathsf{T} \vdash \mathrm{TI}(t, Q)$, according to Definition 3.4, the ordinal represented by t is provable in T. When working with $\prec$, we adhere to the notation common with ordinals. In particular, we use symbols such as 0, 1, ε_0 or $\varphi_1(\varepsilon_0)$ for designated ordinal terms representing the corresponding ordinal. Moreover, following [Buc05], there are p.r. functions $\cdot \hat{+} \cdot$, $\omega^{\cdot}$ and $\varphi_{\cdot}(\cdot)$ representing ordinal addition, taking powers with base ω, and the binary Veblen functions on $\mathrm{field}(\preceq)$. Finally, we note that every $\alpha \in \mathrm{field}(\preceq$ $)$ can be written in Cantor normal form, i.e.,

$$\alpha = \omega^{\alpha_1} \hat{+} \ldots \hat{+} \omega^{\alpha_m}$$

for unique $\alpha_1, \ldots, \alpha_m$, where $m \geq 1$. There exists binary p.r. function symbols h and e such that

$$\mathrm{h}(\alpha) = \omega^{\alpha_1} \hat{+} \ldots \hat{+} \omega^{\alpha_{m-1}} \quad \text{and} \quad \mathrm{e}(\alpha) = \alpha_m.$$

One can then define the p.r. function $\hat{\varphi}_{\cdot}(\cdot)$ satisfying

$$\hat{\varphi}_\alpha(\beta) = \varphi_{\alpha_1}(\varphi_{\alpha_2}(\cdots(\varphi_{\alpha_m}(\beta))\cdots)).$$

This function will be important in the next section.

3.3. The systems FP_0^- and $\mathsf{pr\text{-}ATR}_0^-$

As shown in the previous section, removing set-parameters does not affect ATR_0 and ATR. We now turn to a set-parameter free variant of FP_0, namely FP_0^-, and pin down a proof-theoretically equivalent variant of ATR_0^-, denoted $\mathsf{pr\text{-}ATR}_0^-$, where transfinite recursion is restricted along initial segments of $\prec$. The effect of adding full induction will be examined in the next section.

It turns out that $|\mathsf{FP}_0^-| = |\mathsf{pr\text{-}ATR}_0^-| = \varphi_{\varepsilon_0}(0)$. First, we establish that FP_0^- is a conservative extension of $\widehat{\mathsf{ID}}_1$, and therefore has the same proof-theoretic strength. Feferman showed that $\widehat{\mathsf{ID}}_1 = \varphi_{\varepsilon_0}(0)$, cf. [Fef82]. Following [Avi96], we proceed by showing that (pr-ATR$^-$) is derivable in FP^-. The section is concluded by showing that $\varphi_{\varepsilon_0}(0) \leq |\mathsf{pr\text{-}ATR}_0^-|$ and $\varphi_{\varphi_1(\varepsilon_0)}(0) \leq |\mathsf{pr\text{-}ATR}^-|$, where $\leq$ denotes the usual order relation on the ordinals. These results will also be used in the next section.

We proceed with stating the exact definitions of the formal systems of interest.

Arithmetical transfinite recursion along the p.r. well ordering $\prec$

Let $\prec$ be the p.r. well ordering fixed in section 3.2. For any number variable α and formula $\mathscr{A}(n, j, X)$ we set

$$\mathcal{H}_{\mathscr{A}}(\alpha, Y) :\equiv Y = \left\{\langle n, \xi\rangle : \xi \prec \alpha \wedge \mathscr{A}(n, \xi, (Y)^{\prec\xi})\right\},$$

where $(Y)^{\prec\xi} := \{\langle m, \mu\rangle \in Y : \mu \prec \xi\}$. The axiom schema (pr-ATR$^-$) consists of all formulas

$$\mathrm{WO}(\alpha) \to \exists Y \mathcal{H}_{\mathscr{A}}(\alpha, Y), \qquad \text{(pr-ATR}^-\text{)}$$

for any number variable α and arithmetical formula $\mathscr{A}(n, j, X)$ with only free set variable X. The system $\mathsf{pr\text{-}ATR}_0^-$ denotes ACA_0 together with (pr-ATR$^-$). $\mathsf{pr\text{-}ATR}^-$ is obtained by adding induction for all $\mathcal{L}_2$ formulas to $\mathsf{pr\text{-}ATR}_0^-$.

Positive arithmetical fixed points without set parameters

The axiom schema (FP$^-$) consists of all formulas

$$\exists X \forall n (n \in X \leftrightarrow \mathscr{A}(n, X)), \tag{FP$^-$}$$

where $\mathscr{A}(n, X)$ is arithmetical with X occurring only positively. Moreover, no set variable besides X is allowed to occur freely in $\mathscr{A}(n, X)$. FP_0^- is ACA_0 plus (FP$^-$). FP^- denotes FP_0^- with induction for all $\mathcal{L}_2$ formulas.

As mentioned, we use $\widehat{\mathsf{ID}}_1$ to establish the proof-theoretic ordinal of FP_0^-. Recall that we write $\mathscr{F}[\vec{x}, \vec{X}]$ to indicate that $\mathscr{F}(\vec{x}, \vec{X})$ is a formula with only the indicated variables occurring freely. For any X-positive arithmetical formula $\mathscr{A}[n, X]$, we let $P_{\mathscr{A}}$ be a unary relation symbol. The language $\mathcal{L}_1^{\mathsf{FP}}$ is obtained from $\mathcal{L}_1$ by adding all these newly defined relation symbols $P_{\mathscr{A}}$ for all $\mathscr{A}[n, X]$ as above. The schema ($\widehat{\mathsf{ID}}_1$) contains all formulas of the form

$$\forall n (P_{\mathscr{A}}(n) \leftrightarrow \mathscr{A}(n, \{x : P_{\mathscr{A}}(x)\})) \tag{$\widehat{\mathsf{ID}}_1$}$$

for all X-positive arithmetical formulas $\mathscr{A}[n, X]$. $\widehat{\mathsf{ID}}_1$ is the formal system in $\mathcal{L}_1^{\mathsf{FP}}$ consisting of PA together with the schema ($\widehat{\mathsf{ID}}_1$), and induction for all $\mathcal{L}_1^{\mathsf{FP}}$ formulas. In the following, $\mathscr{A}(n, P_{\mathscr{A}})$ stands for $\mathscr{A}(n, \{x : P_{\mathscr{A}}(x)\})$. ($\widehat{\mathsf{ID}}_1$) asserts that $P_{\mathscr{A}}$ is a fixed point of the formula $\mathscr{A}[n, X]$. It is known that $|\widehat{\mathsf{ID}}_1| = \varphi_{\varepsilon_0}(0)$, cf. [Fef82]. A simple model-theoretic argument shows that FP_0^- is the second order version of $\widehat{\mathsf{ID}}_1$. Thus, in particular $|\mathsf{FP}_0^-| = \varphi_{\varepsilon_0}(0)$. Before proving this, we introduce an auxiliary notation.

Definition 3.5.

(a) Let $\mathcal{L}$ be a first order language with $\mathcal{L}_1 \subseteq \mathcal{L}$. If $\mathcal{M}$ is a structure for $\mathcal{L}$ and $\mathcal{A} \subseteq |\mathcal{M}|$, then $\mathcal{L}(\mathcal{A})$ is the extension of $\mathcal{L}$ with a fresh constant $\underline{c}$ for every $c \in \mathcal{A}$. $\mathcal{M}$ can then be considered as a structure for $\mathcal{L}(\mathcal{A})$ by interpreting $\underline{c}$ as c for all $c \in \mathcal{A}$.

(b) Let $\mathcal{L}$ be a second order language with $\mathcal{L}_2 \subseteq \mathcal{L}$. If $\mathcal{M}$ is a structure for $\mathcal{L}$ and $\mathcal{A} \subseteq |\mathcal{M}| \cup \mathcal{S}^{\mathcal{M}}$, then $\mathcal{L}(\mathcal{A})$ is the extension of $\mathcal{L}$ with a fresh constant $\underline{c}$ for every $c \in \mathcal{A}$. $\mathcal{M}$ then can be considered as a structure for $\mathcal{L}(\mathcal{A})$ by interpreting $\underline{c}$ as c for all $c \in \mathcal{A}$.

Lemma 3.6. *Every model of* $\widehat{\mathsf{ID}}_1$ *can be extended to a model of* FP_0^-*. Conversely, every model of* FP_0^- *can be reduced to a model of* $\widehat{\mathsf{ID}}_1$*. In both cases, the interpretation of* $\mathcal{L}_1$ *formulas does not change.*

Proof. We start by showing that every model of $\widehat{\mathsf{ID}}_1$ can be extended to a model of FP_0^-. Consider a model $\mathcal{M}$ of $\widehat{\mathsf{ID}}_1$. Let $\mathcal{S}$ be the collection of all sets that are definable by $\mathcal{L}_1^{\mathsf{FP}}(|\mathcal{M}|)$ formulas. This means that $A \in \mathcal{S}$ iff there is some $\mathcal{L}_1^{\mathsf{FP}}(|\mathcal{M}|)$ formula $\mathscr{F}[n]$ such that

$$A = \{n : \mathcal{M} \vDash \mathscr{F}[n]\}.$$

Given a set $A \in \mathcal{S}$, we write $\mathscr{F}_A[n]$ to denote its defining formula with only free variable n. Adding $\mathcal{S}$ to $\mathcal{M}$ results in an $\mathcal{L}_2^{\mathsf{FP}}$ structure. We show that this structure is a model of FP_0^-.

Concerning (ACA), let $\mathscr{A}[n]$ be an arithmetical formula in $\mathcal{L}_2(|\mathcal{M}| \cup \mathcal{S}^{\mathcal{M}})$. We substitute every set constant $Z \in \mathcal{S}$ occurring in $\mathscr{A}[n]$ with $\{x : \mathscr{F}_Z(x)\}$. The result of this procedure is a $\mathcal{L}_1^{\mathsf{FP}}(|\mathcal{M}|)$ formula $\mathscr{B}[n]$. The corresponding set of $\mathcal{S}$ defined by $\mathscr{B}[n]$ is a proper witness for the given instance of (ACA).

For (FP^-) we consider an X-positive arithmetical formula $\mathscr{A}[n, \vec{z}, X]$ in the language $\mathcal{L}_2$ with only the indicted variables occurring freely. We define

$$\mathscr{A}^*[m, X] = \exists n, \vec{z}(m = \langle n, \vec{z}\rangle \wedge \mathscr{A}[n, \vec{z}, X]).$$

Since $\mathcal{M}$ is a model of $\widehat{\mathsf{ID}}_1$, we have

$$\forall m(P_{\mathscr{A}^*}(m) \leftrightarrow \exists n, \vec{z}(m = \langle n, \vec{z}\rangle \wedge \mathscr{A}(n, \vec{z}, P_{\mathscr{A}^*}))).$$

Substituting $\vec{z}$ with a matching list of constants $\vec{c}$ from $|\mathcal{M}|$, we can deduce by the above that

$$\forall n(P_{\mathscr{A}^*}(\langle n, \vec{c}\rangle) \leftrightarrow \mathscr{A}(n, \vec{c}, P_{\mathscr{A}^*})).$$

By definition $\{n : P_{\mathscr{A}^*}(\langle n, \vec{c}\rangle)\}$ is a set of $\mathcal{S}$, and, as shown above, a proper witness for the instance of FP_0^- with formula $\mathscr{A}[n, \vec{c}, X]$.

Finally, since induction for all $\mathcal{L}_1^{\mathsf{FP}}(|\mathcal{M}|)$ formulas is valid in $\mathcal{M}$, and $\mathcal{S}$ consists exactly of the sets definable by such formulas, the induction axiom is valid in $\mathcal{M}$.

For the converse direction, let $\mathcal{N}$ be a model of FP_0^-. By (FP^-), there exists a set $Z_{\mathscr{A}}$ in $\mathcal{S}^{\mathcal{N}}$ such that $Z_{\mathscr{A}} = \{n : \mathscr{A}[n, Z_{\mathscr{A}}]\}$ for every X-positive arithmetical formula $\mathscr{A}[n, X]$. We can now transform $\mathcal{N}$ into a $\mathcal{L}_1^{\mathsf{FP}}$ structure $\mathcal{N}^+$ by deleting the second order part of $\mathcal{N}$, and interpreting the relation symbol $P_{\mathscr{A}}^{\mathcal{N}}$ by the set $Z_{\mathscr{A}}$. Obviously, ($\widehat{\mathsf{ID}}_1$) is valid in $\mathcal{N}^+$.

To deal with induction for $\mathcal{L}_1^{\mathsf{FP}}$ formulas, let, e.g., $\mathscr{B}[n, P_{\mathscr{A}_1}, P_{\mathscr{A}_2}]$ be a $\mathcal{L}_1^{\mathsf{FP}}(|\mathcal{N}^+|)$ formula with only free variable n. By assumption, induction for $\mathscr{B}[n, Z_{\mathscr{A}_1}, Z_{\mathscr{A}_2}]$ is available in $\mathcal{N}$. Thus, by construction, induction for $\mathscr{B}[n, P_{\mathscr{A}_1}, P_{\mathscr{A}_2}]$ holds in $\mathcal{N}^+$. This finishes the proof. □

As mentioned we have the following theorem proven in [Fef82], from which $|\mathsf{FP}_0^-|$ can be immediately determined by Lemma 3.6.

Theorem 3.7 (Feferman)**.** $|\widehat{\mathsf{ID}}_1| = \varphi_{\varepsilon_0}(0)$.

Corollary 3.8. $|\mathsf{FP}_0^-| = \varphi_{\varepsilon_0}(0)$.

We proceed with deriving (pr-ATR$^-$) in FP_0^-. It turns out that Avigad's proof of the reverse direction in [Avi96] can not be adapted to prove that $\mathsf{pr\text{-}ATR}_0^-$ proves FP_0^-. The proof relies on deriving a fixed point using a pseudohierarchy, which is not clear how to do in $\mathsf{pr\text{-}ATR}_0^-$.

Lemma 3.9. *The schema* (pr-ATR$^-$) *is provable in* FP_0^-.

Proof. Suppose we are working in FP_0^-. Let $\mathscr{A}(n, \xi, X)$ be an arithmetical formula with only X as free set variable and assume $\mathrm{WO}(\alpha)$. Note that by our definition of formulas, $\mathscr{A}(n, \xi, X)$ is in negation normal form. Given a set X and number ξ we let

$$\begin{aligned}
[X]^1_\xi &:= \{m : \exists n, \mu(m = \langle n, \mu\rangle \wedge \mu \prec \xi \wedge \langle m, 1\rangle \in X)\}\,, \\
[X]^0_\xi &:= \{m : \forall n, \mu(m = \langle n, \mu\rangle \to (\mu \not\prec \xi \vee \langle m, 0\rangle \in X))\}\,.
\end{aligned}$$

By (ACA), these exist properly as sets. Next, we modify $\mathscr{A}(n, \xi, X)$ in two steps. First, we consider a fresh set variable U and replace all subformulas in $\mathscr{A}(n, \xi, X)$ of the form $t \notin X$ by $t \in U$. Let $\mathscr{A}^*(n, \xi, X, U)$ denote the resulting formula. Then we define

$$\mathscr{A}^+(n, \xi, X) :\equiv \mathscr{A}^*(n, \xi, [X]^1_\xi, [X]^0_\xi).$$

The above procedure can also be applied to $\neg\mathscr{A}(n, \xi, X)$. This results in the formula $(\neg\mathscr{A})^+(n, \xi, X)$. Observe that X occurs only positively in

$\mathscr{A}^+(n,\xi,X)$ and $(\neg\mathscr{A})^+(n,\xi,X)$. As a next step, we define the formula

$$\begin{aligned}\mathscr{B}(z,X) :\equiv \exists n,\xi,w(z = \langle\langle n,\xi\rangle,w\rangle \wedge \xi \prec \alpha \wedge \\ ((w=1 \wedge \mathscr{A}^+(n,\xi,X)) \vee (w=0 \wedge (\neg\mathscr{A})^+(n,\xi,X)))).\end{aligned}$$

Applying (FP$^-$) to $\mathscr{B}(z,X)$ yields a set Z such that

$$Z = \{z : \mathscr{B}(z,Z)\}.$$

We will proof that Z is a function assigning 0 or 1 to all pairs $\langle n,\xi\rangle$ with $\xi \prec \alpha$. This will be achieved by using transfinite induction to show that for all $\xi \prec \alpha$:

$$\forall n(\langle\langle n,\xi\rangle,0\rangle \in Z \leftrightarrow \langle\langle n,\xi\rangle,1\rangle \notin Z).$$

We first observe that by definition of Z, we have that $\langle\langle n,\xi\rangle,w\rangle \in Z$ iff

$$(w=1 \wedge \mathscr{A}^+(n,\xi,Z)) \vee (w=0 \wedge (\neg\mathscr{A})^+(n,\xi,Z)).$$

Let $\mu \prec \xi$. By the induction hypothesis we get

$$\begin{aligned}\langle n,\mu\rangle \notin [Z]^1_\xi &\leftrightarrow i \not\prec \xi \vee \langle\langle n,\mu\rangle,1\rangle \notin Z \\ &\leftrightarrow \mu \not\prec \xi \vee \langle\langle n,\mu\rangle,0\rangle \in Z \\ &\leftrightarrow \langle n,\mu\rangle \in [Z]^0_\xi.\end{aligned}$$

By induction on the build-up of $\mathscr{A}(n,\xi,Z)$ we get that either $\mathscr{A}^+(n,\xi,Z)$ or $(\neg\mathscr{A})^+(n,\xi,Z)$ holds, but not both. If $\mathscr{A}(n,\xi,Z)$ is of the form $t \in Z$, where t is some number term, the claim follows by the above since we have

$$\begin{aligned}(t \in Z)^+ &\equiv \exists n,\mu(t = \langle n,\mu\rangle \wedge \mu \prec \xi \wedge \langle t,1\rangle \in Z),\\ (t \notin Z)^+ &\equiv \forall n,\mu(t = \langle n,\mu\rangle \to (\mu \not\prec \xi \vee \langle t,0\rangle \in Z))\\ &\equiv \forall n,\mu(t = \langle n,\mu\rangle \to (\mu \not\prec \xi \vee \langle t,1\rangle \notin Z)).\end{aligned}$$

The other cases follow purely by our underlying logic and are straightforward to verify. This concludes the induction on the build-up. It follows that $w = 0$ or $w = 1$, but not both, hence the transfinite induction is finished. Thus, Z is a characteristic function as claimed.

Next, we observe that for $\xi \prec \alpha$ it holds that $[Z]^1_\xi = ([Z]^1_\alpha)^{\prec\xi}$ and $[Z]^0_\xi = \overline{([Z]^1_\alpha)^{\prec\xi}}$, where for any set U, $\overline{U}$ denotes its complement, i.e.,

$\overline{U} = \{x : x \notin U\}$. The first assertion follows since:

$$\begin{aligned}
[Z]^1_\xi &= \{m : \exists n, \mu(m = \langle n, \mu\rangle \wedge \mu \prec \xi \wedge \langle m, 1\rangle \in Z)\} \\
&= \{\langle n, \mu\rangle : \mu \prec \xi \wedge \langle\langle n, \mu\rangle, 1\rangle \in Z\} \\
&= \left\{\langle n, \mu\rangle \in [Z]^1_\alpha : \mu \prec \xi\right\} \\
&= ([Z]^1_\alpha)^{\prec\xi}.
\end{aligned}$$

For the second assertion we make a case distinction. Suppose m is not a pair, i.e., $\forall x, y(m \neq \langle x, y\rangle)$. Then clearly

$$m \in [Z]^0_\xi \leftrightarrow m \in \overline{([Z]^1_\alpha)^{\prec\xi}}.$$

If m is the pair $\langle n, \mu\rangle$, it follows that

$$\begin{aligned}
m \in [Z]^0_\xi &\leftrightarrow (\mu \not\prec \xi \vee \langle m, 0\rangle \in Z) \\
&\leftrightarrow (\mu \not\prec \xi \vee \langle m, 1\rangle \notin Z) \\
&\leftrightarrow m \in \overline{[Z]^1_\xi} \\
&\leftrightarrow m \in \overline{([Z]^1_\alpha)^{\prec\xi}},
\end{aligned}$$

where we used the first assertion, i.e., $[Z]^1_\xi = ([Z]^1_\alpha)^{\prec\xi}$, and that Z is a characteristic function. This establishes the second assertion. Looking back at the definition of $\mathscr{B}(z, Z)$ we can now deduce that

$$\begin{aligned}
[Z]^1_\alpha &= \{\langle n, \xi\rangle : \xi \prec \alpha \wedge \langle\langle n, \xi\rangle, 1\rangle \in Z\} \\
&= \left\{\langle n, \xi\rangle : \xi \prec \alpha \wedge \mathscr{A}^+(n, \xi, Z)\right\} \\
&= \left\{\langle n, \xi\rangle : \xi \prec \alpha \wedge \mathscr{A}^*(n, \xi, [Z]^1_\xi, [Z]^0_\xi)\right\} \\
&= \left\{\langle n, \xi\rangle : \xi \prec \alpha \wedge \mathscr{A}^*(n, \xi, ([Z]^1_\alpha)^{\prec\xi}, \overline{([Z]^1_\alpha)^{\prec\xi}})\right\} \\
&= \left\{\langle n, \xi\rangle : \xi \prec \alpha \wedge \mathscr{A}(n, \xi, ([Z]^1_\alpha)^{\prec\xi})\right\}.
\end{aligned}$$

The above means that $\mathcal{H}_{\mathscr{A}}(\alpha, [Z]^1_\alpha)$. Thus, $[Z]^1_\alpha$ is the desired set and the proof is finished. $\square$

Corollary 3.10. *It holds that*

$$|\mathsf{pr\text{-}ATR}_0^-| \leq |\mathsf{FP}_0^-| \quad \textit{and} \quad |\mathsf{pr\text{-}ATR}^-| \leq |\mathsf{FP}^-|.$$

Next, lower bounds for the proof-theoretic strength of $\mathsf{pr\text{-}ATR_0^-}$ and $\mathsf{pr\text{-}ATR^-}$ will be established. The proof relies on methods given in [Buc05]. The following formulas will be needed:

$$\begin{aligned}
\mathrm{Sp}(X) &:\equiv \left\{\beta : (\forall \xi \subseteq X)(\xi \hat{+} \beta \subseteq X)\right\}, \\
\mathrm{Sp}^*(\alpha, X) &:\equiv \{\beta : (\forall \xi \prec \alpha)(\varphi(\mathrm{e}(\alpha), \beta) \in \mathrm{Sp}((X)_\xi))\}, \\
\mathscr{R}(\alpha, X) &:\equiv (\forall \xi \preceq \alpha)(0 \prec \xi \rightarrow (Y)_\xi = \mathrm{Sp}^*(\xi, Y)), \\
\mathscr{R}(\alpha) &:\equiv \exists Y((Y)_0 = Q \wedge \mathscr{R}(\alpha, Y)).
\end{aligned}$$

The fact shown below will be useful later. It follows immediately by observing that for all $\xi \prec \alpha$, $(X)_\xi = ((X)^{\prec \alpha})_\xi$:

$$\mathrm{Sp}^*(\alpha, X) = \mathrm{Sp}^*(\alpha, (X)^{\prec \alpha}). \tag{3.1}$$

Moreover, we will make use of the following theorem, taken from [Buc05].

Theorem 3.11. *It is provable in* $\mathsf{ACA_0}$ *that*

$$\mathrm{WO}(\alpha) \wedge \mathscr{R}(\alpha) \rightarrow \mathrm{TI}(\hat{\varphi}_\alpha(0), Q).$$

Next, we define the formulas

$$\begin{aligned}
\mathscr{F}(n, \xi, X) &:\equiv (\xi = 0 \rightarrow n \in Q) \wedge (\xi \neq 0 \rightarrow n \in \mathrm{Sp}^*(\xi, X)), \\
\tilde{\mathcal{H}}_{\mathscr{F}}(\alpha, Y) &:\equiv (\forall \xi \preceq \alpha)((Y)_\xi = \left\{n : \mathscr{F}(n, \xi, (Y)^{\prec \xi})\right\}).
\end{aligned}$$

Lemma 3.12. $\mathsf{ACA_0}$ *proves* $\tilde{\mathcal{H}}_{\mathscr{F}}(\alpha, Y) \leftrightarrow ((Y)_0 = Q \wedge \mathscr{R}(\alpha, Y))$.

Proof. By the definition of $\mathscr{F}(n, \xi, X)$ we have

- $\left\{n : \mathscr{F}(n, 0, (Y)^{\prec 0})\right\} = Q$,

and for $\xi \succ 0$,

- $\left\{n : \mathscr{F}(n, \xi, (Y)^{\prec \xi})\right\} = \mathrm{Sp}^*(\xi, (Y)^{\prec \xi}) = \mathrm{Sp}^*(\xi, Y)$,

where the last equation holds because of (3.1). From that the assertion follows immediately. □

Corollary 3.13. $\mathsf{ACA_0}$ *proves* $\exists Y \tilde{\mathcal{H}}_{\mathscr{F}}(\alpha, Y) \leftrightarrow \mathscr{R}(\alpha)$.

Lemma 3.14. *Working in* $\mathsf{pr\text{-}ATR}_0^-$*, we can prove that for all* α

$$\mathrm{WO}(\alpha) \to \mathrm{TI}(\hat{\varphi}_\alpha(0), Q).$$

Proof. Let us work in $\mathsf{pr\text{-}ATR}_0^-$ and assume $\mathrm{WO}(\alpha)$. Using (ACA), it follows that $\mathrm{WO}(\alpha \hat{+} 1)$. Hence, by (pr-ATR$^-$), there exists a set Y with $\mathcal{H}_{\mathscr{F}}(\alpha \hat{+} 1, Y)$. Clearly, Y also satisfies $\tilde{\mathcal{H}}_{\mathscr{F}}(\alpha, Y)$, therefore by Corollary 3.13, we have $\mathscr{R}(\alpha)$. The claim now follows by Theorem 3.11. $\square$

The following are standard results from proof theory and will allow us to establish the lower bounds for $\mathsf{pr\text{-}ATR}_0^-$ and $\mathsf{pr\text{-}ATR}^-$.

Theorem 3.15. *We have for all ordinal terms a:*

(a) $\mathsf{ACA}_0 \vdash \mathrm{WO}(a)$ *if* $a \prec \varepsilon_0$,

(b) $\mathsf{ACA} \vdash \mathrm{WO}(a)$ *if* $a \prec \varphi_1(\varepsilon_0)$.

Theorem 3.16.

(a) $\varphi_{\varepsilon_0}(0) \leq |\mathsf{pr\text{-}ATR}_0^-|$.

(b) $\varphi_{\varphi_1(\varepsilon_0)}(0) \leq |\mathsf{pr\text{-}ATR}^-|$.

Proof. To show (a), let a be an ordinal term with $a \prec \varphi_{\varepsilon_0}(0)$. Then there exists an ordinal term $b \prec \varepsilon_0$ such that $a \prec \hat{\varphi}_b(0)$. In view of Theorem 3.15 and Lemma 3.14 we have that

$$\mathsf{pr\text{-}ATR}_0^- \vdash \mathrm{TI}(\hat{\varphi}_b(0), Q),$$

and, consequently, $\mathsf{pr\text{-}ATR}_0^- \vdash \mathrm{TI}(a, Q)$. This proves (a).

For (b) we proceed analogously. Now, consider an ordinal term a with $a \prec \varphi_{\varphi_1(\varepsilon_0)}(0)$. There exists an ordinal term $b \prec \varphi_1(\varepsilon_0)$ such that $a \prec \hat{\varphi}_b(0)$. We apply Theorem 3.15 and Lemma 3.14, and obtain

$$\mathsf{pr\text{-}ATR}^- \vdash \mathrm{TI}(\hat{\varphi}_b(0), Q).$$

Therefore, $\mathsf{pr\text{-}ATR}_0^- \vdash \mathrm{TI}(a, Q)$, and we are done. $\square$

Combing all constituents, we arrive at the final result of this section.

Corollary 3.17. *The systems* $\mathsf{pr\text{-}ATR}_0^-$ *and* FP_0^- *both have proof-theoretic strength* $\varphi_{\varepsilon_0}(0)$.

Proof. Recall that Corollary 3.8 says that $|\mathsf{FP}_0^-| = \varphi_{\varepsilon_0}(0)$. Moreover, Corollary 3.10 and Theorem 3.16 yield

$$\varphi_{\varepsilon_0}(0) \leq |\mathsf{pr\text{-}ATR}_0^-| \leq |\mathsf{FP}_0^-|.$$

This proves the assertion. □

3.4. The systems FP^- and $\mathsf{pr\text{-}ATR}^-$

Recall that the systems FP^- and $\mathsf{pr\text{-}ATR}^-$ are obtained from, respectively, FP_0^- and $\mathsf{pr\text{-}ATR}^-$ by adding $\mathcal{L}_2$ induction. We now determine the proof-theoretic strength of these. It turns out that $|\mathsf{FP}^-| = |\mathsf{pr\text{-}ATR}^-| = \varphi_{\varphi_1(\varepsilon_0)}(0)$. By Corollary 3.10 and Theorem 3.16, it suffices to show that $|\mathsf{FP}^-| \leq \varphi_{\varphi_1(\varepsilon_0)}(0)$. This follows in a straightforward manner from results in [JS95]. To carry out the details, the systems $\mathsf{FP}^\ominus$ and $\widehat{\mathsf{EID}_1}$ are introduced. Recall that $\mathscr{F}[\vec{x}, \vec{X}]$ signifies that no variables other than the ones displayed occur freely in $\mathscr{F}(\vec{x}, \vec{X})$. The schema ($\mathrm{FP}^\ominus$) consists of all formulas of the form

$$\exists X \forall n (n \in X \leftrightarrow \mathscr{A}[n, X]) \qquad (\mathrm{FP}^\ominus)$$

for any X-positive arithmetical formula $\mathscr{A}[n, X]$. $\mathsf{FP}^\ominus$ denotes ACA plus ($\mathrm{FP}^\ominus$).

Lemma 3.18. *Over* ACA_0, ($\mathrm{FP}^\ominus$) *and* (FP^-) *are equivalent.*

Proof. Suppose we are working in ACA_0. It suffices to derive (FP^-) using ($\mathrm{FP}^\ominus$). Let $\mathscr{A}[n, \vec{m}, X]$ be an X-positive arithmetical formula, where $\vec{m} = m_1, \ldots, m_l$, $l \geq 0$. For any set Y, put $(Y)_{\vec{m}} := \{n : \langle n, \vec{m} \rangle \in Y\}$ and define

$$\mathscr{B}[k, X] :\equiv \exists n, \vec{m}(k = \langle n, \vec{m} \rangle \wedge \mathscr{A}(n, \vec{m}, (Y)_{\vec{m}})).$$

Clearly, ($\mathrm{FP}^\ominus$) is applicable to $\mathscr{B}[k, X]$, yielding a set Z such that $Z = \{k : B[k, X]\}$. It follows that

$$\begin{aligned}(Z)_{\vec{m}} &= \{n : \langle n, \vec{m} \rangle \in Z\} \\ &= \{n : \mathscr{A}(n, \vec{m}, (Z)_{\vec{m}})\},\end{aligned}$$

hence $(Z)_{\vec{m}}$ is a fixed point for $\mathscr{A}[n, \vec{m}, X]$. This establishes ($\mathrm{FP}^-$). □

In order to define $\widehat{\mathsf{EID}_1}$, we extend $\mathcal{L}_2$ with unary relation symbols $P_{\mathscr{A}}$ for all X-positive arithmetical formulas $\mathscr{A}[n, X]$. The resulting language is denoted $\mathcal{L}_2^{\mathsf{FP}}$. An $\mathcal{L}_2^{\mathsf{FP}}$ formula is called elementary if no bound set variables occur in it. Note that this does not restrict the occurrence of free set variables and relation symbols $P_{\mathscr{A}}$. $\widehat{\mathsf{EID}_1}$ is the formal system in the language $\mathcal{L}_2^{\mathsf{FP}}$ consisting of PA, induction for all $\mathcal{L}_2^{\mathsf{FP}}$ formulas, and elementary comprehension, i.e,

$$\exists X \forall n (n \in X \leftrightarrow \mathscr{A}(n)) \tag{ECA}$$

for all elementary $\mathcal{L}_2^{\mathsf{FP}}$ formulas $\mathscr{A}(n)$ with $X \notin \mathrm{FrVar}(\mathscr{A})$. Moreover, $\widehat{\mathsf{EID}_1}$ contains the schema ($\widehat{\mathsf{ID}}_1$), defined in section 3.3, i.e.,

$$\forall n (P_{\mathscr{A}}(n) \leftrightarrow \mathscr{A}(n, P_{\mathscr{A}})) \tag{$\widehat{\mathsf{ID}}_1$}$$

for all X-positive arithmetical $\mathcal{L}_2$ formulas $\mathscr{A}[n, X]$.

Corollary 3.19. FP^- *is a subsytem of* $\widehat{\mathsf{EID}_1}$.

Proof. Clearly, $\mathsf{FP}^\ominus \subseteq \widehat{\mathsf{EID}_1}$ since ($\widehat{\mathsf{ID}}_1$) implies ($\mathrm{FP}^\ominus$) and all other axioms of $\mathsf{FP}^\ominus$ are also axioms of $\widehat{\mathsf{EID}_1}$. The claim now follows by Lemma 3.18. □

From [JS95] we get the following theorem, which together with Corollary 3.19 gives the desired upper bound for FP_0^-.

Theorem 3.20 (Jäger, Strahm)**.** $|\widehat{\mathsf{EID}_1}| = \varphi_{\varphi_1(\varepsilon_0)}(0)$.

Corollary 3.21. $|\mathsf{FP}^-| \leq \varphi_{\varphi_1(\varepsilon_0)}(0)$.

We conclude by stating the main result of this section.

Corollary 3.22. *The systems* $\mathsf{pr\text{-}ATR}^-$ *and* FP^- *both have proof-theoretic strength* $\varphi_{\varphi_1(\varepsilon_0)}(0)$.

Proof. Corollary 3.10, Theorem 3.16 and Corollary 3.21 yield

$$\varphi_{\varphi_1(\varepsilon_0)}(0) \leq |\mathsf{pr\text{-}ATR}^-| \leq |\mathsf{FP}^-| \leq \varphi_{\varphi_1(\varepsilon_0)}(0).$$

Thus, the assertion is established. □

Part II.

Subsystems of Set Theory

4. Preliminaries

In Part I several principles equivalent to (ATR) were introduced. It is the goal of Part II to study analogous principles in set-theoretic contexts. For reference we refer to [Sim09, Bar75, BJ20]. Our set theory features natural numbers as urelements and will be introduced in this section. Later we will turn to stronger set theories featuring, among other principles, the Axiom Beta. Our approach will be equivalent to the one chosen in [Sim09].

4.1. The language $\mathcal{L}_\mathrm{s}$

In section 1.1 we introduced the second order language $\mathcal{L}_2$. We now turn to the set-theoretic *language* $\mathcal{L}_\mathrm{s}$. In subsequent sections we will often switch between $\mathcal{L}_\mathrm{s}$ and $\mathcal{L}_2$. To avoid ambiguities we will speak contextually, e.g., of $\mathcal{L}_\mathrm{s}$ formulas, and so on. However, if clear from context we omit mentioning the language. The first order language $\mathcal{L}_\mathrm{s}$ comprises of the following *logical symbols*:

- *(propositional) connectives* $\wedge$ and $\vee$,
- the *negation symbol* $\sim$ to form negated literals,
- the *quantifier symbols* $\forall$ and $\exists$,
- the *set membership symbol* $\in$,
- the *equality symbol* $=_\mathrm{N}$,
- a countable set of *object variables* $\mathrm{Var}_s := \{\chi_i : i \in \mathbb{N}\}$,
- auxiliary symbols, e.g., comma and parentheses.

On the non-logical side we use the following symbols:

- the *constant symbol* N,
- a constant $\underline{n}$ for each (standard) natural number $n \in \mathbb{N}$,

- the *unary relation symbol* S,
- *symbols* for all *p.r. relations*,
- a *unary anonymous relation symbol* Q,
- additional *relation symbols* introduced contextually.

We consider set theories above the natural numbers as urelements. N is a set constant for the set of all natural numbers and the relation symbol S singles out those objects that are sets. Hence, we have $\mathrm{S}(x)$ iff x is not an element of N. The equality symbol $=_{\mathrm{N}}$ serves to identify equal urelements. The equality of arbitrary objects will be defined later.

We have relation symbols for all p.r. functions and a constant $\underline{n}$ for any natural number n. In particular, $=_{\mathrm{N}}$ will coincide with the p.r. equality relation on N. Function symbols for p.r. functions of arity greater than 0 are not permitted. This restriction simplifies the translation of $\mathcal{L}_{\mathrm{s}}$ into $\mathcal{L}_2$, which will be discussed later.

The set $\mathrm{Tm}(\mathcal{L}_{\mathrm{s}})$ of $\mathcal{L}_{\mathrm{s}}$ terms is defined as follows:

- All object variables and the constant N are terms.
- The constant $\underline{n}$ is a term for all $n \in \mathbb{N}$.

An *atomic formula* (of $\mathcal{L}_{\mathrm{s}}$) is an expression of the form $s =_{\mathrm{N}} t$, $s \in t$ or $R(t_1, \ldots, t_m)$ for any terms $s, t, t_1, \ldots, t_m$ and m-ary relation symbol R. A *literal* is either an atomic formula $\mathscr{A}$ or an expression of the form $\sim\mathscr{A}$, where $\mathscr{A}$ is an atomic formula. We write $s \notin t$ for $\sim(s \in t)$, and so on. *Equality of objects*, which encompasses natural numbers and sets, will be defined separately. Formulas are inductively defined as follows:

- All literals are formulas.
- If $\mathscr{A}$ and $\mathscr{B}$ are formulas, then so are $(\mathscr{A} \wedge \mathscr{B})$, $(\mathscr{A} \vee \mathscr{B})$, $\forall \chi_i \mathscr{A}$, and $\exists \chi_i \mathscr{A}$ for all $i \in \mathbb{N}$.

$\mathrm{Fml}(\mathcal{L}_{\mathrm{s}})$ stands for the set of all $\mathcal{L}_{\mathrm{s}}$ formulas. Recall that $\mathscr{A}(\vec{x})$ stands for a formula with the (object) variables in $\vec{x}$ occurring freely. This does not exclude the occurrence of additional free variables. To exclude additional free variables we write $\mathscr{A}[\vec{x}]$. The following (possibly subscripted) metavariables are used when working in $\mathcal{L}_{\mathrm{s}}$:

- $k, l, m, n, u, v, w, x, y, z$ for object variables,

- q, r, s, t for $\mathcal{L}_s$ terms,
- $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{F}$ for formulas.

If there is no danger of confusion, we simply speak of variables instead of object variables. For better readability, parentheses are used freely in the following. When working in our set theory, we usually write $0, 1, \ldots$ instead of $\underline{0}, \underline{1}$, etc. The negation of a formula $\mathscr{A}$, denoted $\neg\mathscr{A}$, is defined inductively as in section 1.1. This is also the case for the connectives $\rightarrow$ and $\leftrightarrow$. Given a term t and a formula $\mathscr{A}$, the expressions $(\forall x \in t)\mathscr{A}$ and $(\exists x \in t)\mathscr{A}$, stand for, respectively, the formulas $\forall x(x \in t \rightarrow \mathscr{A})$ and $\exists x(x \in t \wedge \mathscr{A})$. Moreover, if t is a term, we write $\mathscr{A}^t$ for the result of replacing all unbounded quantifiers $\exists x(\ldots)$ and $\forall x(\ldots)$ in $\mathscr{A}$ by, respectively, $(\exists x \in t)(\ldots)$ and $(\forall x \in t)(\ldots)$.

Definition 4.1. The class of Δ_0 formulas of $\mathcal{L}_s$ is defined inductively as follows:

- Every literal is a Δ_0 formula.
- If $\mathscr{A}$ and $\mathscr{B}$ are Δ_0, then so are $\mathscr{A} \wedge \mathscr{B}$, $\mathscr{A} \vee \mathscr{B}$, $(\forall x \in t)\mathscr{A}$ and $(\exists x \in t)\mathscr{A}$, where the object variable x must not occur in the term t.

Given $k \in \mathbb{N}$, a formula is called Σ_k if it is of the form $\exists x_1 \forall x_2 \cdots x_k \mathscr{A}$, where $\mathscr{A}$ is Δ_0. Similarly, $\mathscr{A}$ is Π_k if it is of the form $\forall x_1 \exists x_2 \cdots x_k \mathscr{A}$, with $\mathscr{A}$ being Δ_0. Note that the class of Δ_0 formulas is closed under negation. The negation of a Σ_k formula is Π_k, and vice-versa.

Definition 4.2 (*Equality of objects*)**.** Given two terms s, t, equality of s, t is defined as follows:

$$s = t :\equiv \begin{cases} (s \in \mathrm{N} \wedge t \in \mathrm{N} \wedge s =_{\mathrm{N}} t) \vee \\ (\mathrm{S}(s) \wedge \mathrm{S}(t) \wedge (\forall x \in s)(x \in t) \wedge (\forall x \in t)(x \in s)). \end{cases}$$

Thus, two objects s, t are equal iff they are primitive recursively equal natural numbers, or they are sets containing the same elements.

As in the case of $\mathcal{L}_2$, we make use of classes. For any $\mathcal{L}_s$ formula $\mathscr{F}(x)$, the class $M = \{x : \mathscr{F}(x)\}$ is said to exist properly if

$$\exists u(\mathrm{S}(u) \wedge \forall x(x \in u \leftrightarrow \mathscr{F}(x))),$$

with u not occurring freely in $\mathscr{F}(x)$. Given a term t and M as above, we write $t = \{x : \mathscr{F}(x)\}$ or $t = M$ for $\mathrm{S}(t) \wedge \forall x(x \in t \leftrightarrow \mathscr{F}(x))$, where t does not occur freely in $\mathscr{F}(x)$. Given a formula $\mathscr{A}(x)$, $(\forall x \in M)\mathscr{A}(x)$ is short for $\forall x(\mathscr{F}(x) \to \mathscr{A}(x))$, and $(\exists x \in M)\mathscr{A}(x)$ stands for $\exists x(\mathscr{F}(x) \wedge \mathscr{A}(x))$. Moreover, for terms s, t, we write $s \subseteq t$ for $\mathrm{S}(s) \wedge \mathrm{S}(t) \wedge (\forall x \in s)(x \in t)$.

The empty class $\emptyset$ is defined as $\{x : \mathrm{S}(x) \wedge {\sim}\mathrm{S}(x)\}$. In addition, given terms s, t, we consider the following classes:

- $\bigcup s = \{x : (\exists y \in s)(x \in y)\}$,
- $\{s, t\} = \{x : x = s \vee x = t\}$,
- $s \setminus t = \{x : x \in s \wedge x \notin t\}$,
- $s \cap t = \{x : x \in s \wedge x \in t\}$,
- $s \cup t = \bigcup \{s, t\}$,
- $\{s\} = \{s, s\}$,
- $\langle s, t\rangle = \{\{s\}, \{s, t\}\}$,
- $s \times t = \{\langle x, y\rangle : x \in s \wedge y \in t\}$,
- $\mathrm{dom}(s) = \{x : \exists y(\langle x, y\rangle \in s)\}$,
- $\mathrm{rng}(s) = \{y : \exists x(\langle x, y\rangle \in s)\}$,
- $\mathrm{field}(s) = \mathrm{dom}(s) \cup \mathrm{rng}(s)$,
- $s^{-1} = \{\langle x, y\rangle : \langle y, x\rangle \in s\}$,
- $t'(x) = \begin{cases} \text{the unique } y \text{ such that } \langle x, y\rangle \in t & \text{if this exists,} \\ \emptyset & \text{otherwise,} \end{cases}$
- $t \restriction s = \{\langle x, y\rangle \in t : x \in s\}$,
- $t''s = \mathrm{rng}(t \restriction s) = \{y : (\exists x \in s)(\langle x, y\rangle \in t)\}$,
- $\in\restriction s = \{\langle x, y\rangle : x \in y \wedge y \in s\}$.

Our axioms will guarantee that these classes exist properly.

Similarly as for $\mathcal{L}_2$, we assume a classical Hilbert-style system featuring tautologies, equality axioms for $=_{\mathrm{N}}$ restricted to N, and axioms for object quantifiers as *logical axioms*. As inference rules we use *modus ponens*, and axioms for *object quantifiers*, analogous to the rules $(A_1\forall)$ and $(A_1\exists)$, cf. section 1.1.

To conclude this section we briefly discuss the semantics of $\mathcal{L}_s$. A structure $\mathcal{M}$ (in the language $\mathcal{L}_s$) consists of

- a non-empty set $|\mathcal{M}|$, called *(object) universe of* $\mathcal{M}$,
- a non-empty set $\mathrm{S}^{\mathcal{M}} \subseteq |\mathcal{M}|$,
- a binary relation $\in^{\mathcal{M}} \subseteq |\mathcal{M}| \times \mathrm{S}^{\mathcal{M}}$,
- an element $\mathrm{N}^{\mathcal{M}} \in \mathrm{S}^{\mathcal{M}}$,
- an element $\underline{n}^{\mathcal{M}} \in |\mathcal{M}| \setminus \mathrm{S}^{\mathcal{M}}$ for each constant $\underline{n}$,
- an m-ary relation $R^{\mathcal{M}} \subseteq (|\mathcal{M}| \setminus \mathrm{S}^{\mathcal{M}})^m$ for each m-ary relation symbol R.

Moreover, the following condition must be met:

- $\{a \in |\mathcal{M}| : a \in^{\mathcal{M}} \mathrm{N}^{\mathcal{M}}\} = |\mathcal{M}| \setminus \mathrm{S}^{\mathcal{M}}$.

An $\mathcal{M}$-assignment is a mapping

$$\mathrm{V}\colon \mathrm{Var}_s \to |\mathcal{M}|.$$

$\mathrm{V}(x : a)$ denotes V with the modified value $a \in |\mathcal{M}|$ at the argument $x \in \mathrm{Var}_s$. Then *"$\mathscr{A}$ is true for* V *in $\mathcal{M}$"* (or *"$\mathcal{M}$ satisfies $\mathscr{A}$ for* V*"*), written $\mathcal{M}, \mathrm{V} \vDash \mathscr{A}$, is defined as usual, cf. section 1.2.

4.2. Basic set theory

In this section, we introduce the system BS of basic set theory and its two subsystems BS^0 and BS^1. To do so we work in the language $\mathcal{L}_s$. BS is a set theory above the natural numbers as urelements and, accordingly, we have two forms of induction: induction on the natural numbers and $\in$-induction. Even the weakest system BS^0 will contain PA.

Recall that $\mathcal{L}_s$ does not allow function symbols except the constants N and $\underline{n}$ for $n \in \mathrm{N}$. As mentioned before, this is for notational simplicity and does not create a problem. It is well-known that Peano arithmetic PA can be formulated in the first order language with

- constants $\underline{n}$ for all $n \in \mathbb{N}$,
- relation symbols for all primitive recursive relations,

as its only non-logical symbols. We assume that Ax(PA) is a sound and complete axiomatization of PA in this language. We now define the formal system BS. We use $s, t \in \mathrm{N}$ to abbreviate $s \in \mathrm{N} \wedge t \in \mathrm{N}$, and so on. The axioms of BS are grouped as follows:

Number-theoretic axioms:

- For every formula $\mathscr{A}$ which is a universal closure of some element of Ax(PA),
$$\mathscr{A}^{\mathrm{N}} \tag{PA}$$
- Full induction on N: For any $\mathcal{L}_{\mathrm{s}}$ formula $\mathscr{A}(x)$,
$$\begin{aligned}&\mathscr{A}(0) \wedge (\forall x, y \in \mathrm{N})(\mathscr{A}(x) \wedge \mathrm{R}_{\mathrm{Succ}}(x, y) \to \mathscr{A}(y))\\ &\qquad \to (\forall x \in N)\mathscr{A}(x),\end{aligned} \tag{$\mathcal{L}_{\mathrm{s}}$-I_{N}}$$
with $\mathrm{R}_{\mathrm{Succ}}$ being the p.r. successor relation.

Ontological axioms: For all terms $s, t, t_1, \ldots, t_m$:

(O1) $s = t \to (\mathscr{A}(s) \to \mathscr{A}(t))$ for all atomic formulas $\mathscr{A}(x)$,

(O2) $\mathrm{S}(s) \vee \mathrm{S}(t) \to s \neq_{\mathrm{N}} t$,

(O3) $\mathrm{S}(\mathrm{N})$,

(O4) $\underline{n} \in \mathrm{N}$ for all $n \in \mathbb{N}$,

(O5) $\mathrm{S}(t) \leftrightarrow t \notin \mathrm{N}$,

(O6) $t \in \mathrm{N} \to s \notin t$.

(O7) For all $m \in \mathbb{N}$ and m-ary p.r. relation symbols R, except Q:
$$R(t_1, \ldots, t_m) \to t_1, \ldots, t_m \in \mathrm{N}.$$

Set-theoretic axioms:

- For all terms s, t,
$$\exists u(u = \{s, t\}). \tag{Pair}$$

- Axioms of Rudimentary Closure (RC): For any sets u, v, the following classes exist properly as sets: $u \setminus v$, $\bigcup u$, $u \times v$, $\mathrm{dom}(u)$, $\in\restriction u$, u^{-1}, and
$$\begin{gathered}\{\langle k, \langle l, m\rangle\rangle : \langle k, l\rangle \in u \wedge m \in v\}, \\ \{\langle k, \langle m, l\rangle\rangle : \langle k, l\rangle \in u \wedge m \in v\}, \\ \{y : \exists x(x \in u \wedge y = v'' \{x\})\}.\end{gathered}$$

- Full $\in$-induction: For any $\mathcal{L}_\mathrm{s}$ formula $\mathscr{A}(x)$,
$$\forall x((\forall y \in x)\mathscr{A}(y) \rightarrow \mathscr{A}(x)) \rightarrow \forall x \mathscr{A}(x). \tag{$\mathcal{L}_\mathrm{s}$-$\mathrm{I}_\in$}$$

We are especially interested in the subsystems BS^0 and BS^1 of BS, which are defined as follows: BS^1 is obtained from BS by deleting full $\in$-induction. BS^0 is BS^1 with ($\mathcal{L}_\mathrm{s}$-I_N) restricted to Δ_0 formulas. The corresponding induction schema is denoted (Δ_0-I_N).

Note that the empty set $\emptyset$ exists properly by the Axioms of Rudimentary Closure. The constants for the natural numbers play the role of numerals.

Crucially, we can prove Δ_0 separation within BS^0, i.e.,

Theorem 4.3. *For any Δ_0 formula $\mathscr{A}(x)$ and object v, BS^0 proves*
$$\exists u(\mathrm{S}(u) \wedge \forall x(x \in u \leftrightarrow (x \in v \wedge \mathscr{A}(x)))), \tag{Δ_0-Sep}$$
where u does not occur freely in $\mathscr{A}(x)$.

Proof. The assertion requires a technically demanding proof. We omit the details and refer to [Jen72]. □

$\mathcal{L}_2$ formulas are transformed canonically into $\mathcal{L}_\mathrm{s}$ formulas. To do so, consider the following partition of the object variables:
$$\{\chi_{2i} : i \in \mathbb{N}\} \quad \text{and} \quad \{\chi_{2i+1} : i \in \mathbb{N}\}.$$

To any number variable ν_i of $\mathcal{L}_2$, we associate the object variable χ_{2i}. To any set variable $\mathcal{V}_i$ of $\mathcal{L}_2$, we assign χ_{2i+1}. We write x° to denote the object variable assigned this way to the number variable x. Analogously, X° denotes the object variable assigned to the set variable X. Moreover, for any $t[\vec{x}] \in \mathrm{Tm}(\mathcal{L}_2)$ with number variables among $\vec{x} = x_1, \dots, x_n$, we define the $\mathcal{L}_\mathrm{s}$ formula $\mathrm{val}_t[y]$ with the object variable y not occurring in $x_1^\circ, \dots, x_n^\circ$ inductively as follows:

- If t is a number variable x_i of $\mathcal{L}_2$, then $\mathrm{val}_t[y] :\equiv y =_\mathrm{N} t^\circ$.
- If $t = f(s_1, \dots, s_m)$ for some m-ary p.r. function symbol f and $\mathcal{L}_2$ terms $s_1, \dots, s_m$, then

  $$\mathrm{val}_t[y] :\equiv (\exists u_1, \dots, u_m \in \mathrm{N}) \left(\bigwedge_{i=0}^{m} \mathrm{val}_{s_i}[u_i] \land \mathrm{R}_f\,(u_1, \dots, u_m, y)\right),$$

  where R_f is the p.r. relation symbol for the graph of f. Moreover, y must not be among $u_1, \dots, u_m$.

Intuitively, $\mathrm{val}_t[y]$ states that the value of y is t. It takes care of $\mathcal{L}_2$ terms by unwrapping them using the relation symbols R_f. Recall that the function symbols of $\mathcal{L}_2$, i.e., symbols for the p.r. relations and the anonymous relation symbol Q, are also part of $\mathcal{L}_\mathrm{s}$. Finally, the translation of $\mathscr{A} \in \mathrm{Fml}(\mathcal{L}_2)$ into $\mathscr{A}^\circ \in \mathrm{Fml}(\mathcal{L}_\mathrm{s})$ is given as follows:

- $(s = t)^\circ :\equiv (\exists x, y \in \mathrm{N})(\mathrm{val}_s[x] \land \mathrm{val}_t[y] \land x =_\mathrm{N} y)$,
- $(s \in X)^\circ :\equiv (\exists x \in \mathrm{N})(\mathrm{val}_s[x] \land x \in X^\circ)$,
- For all m-ary relation symbols R and $m \in \mathbb{N}$:

  $$R(t_1, \dots, t_m)^\circ :\equiv (\exists x_1, \dots, x_m \in \mathrm{N}) \left(\bigwedge_{i=0}^{m} \mathrm{val}_{t_i}[x_i] \land R(x_1, \dots, x_m)\right).$$

- $(\mathscr{B}_1 \diamond \mathscr{B}_2)^\circ :\equiv \mathscr{B}_1^\circ \diamond \mathscr{B}_2^\circ$ for $\diamond \in \{\land, \lor\}$,
- $(\mathrm{Q}x\mathscr{B})^\circ :\equiv (\mathrm{Q}x^\circ \in \mathrm{N})\mathscr{B}^\circ$ for $\mathrm{Q} \in \{\exists, \forall\}$,
- $(\mathrm{Q}X\mathscr{B})^\circ :\equiv (\mathrm{Q}X^\circ \subseteq \mathrm{N})\mathscr{B}^\circ$ for $\mathrm{Q} \in \{\exists, \forall\}$.

Note that the free (object) variables in $\mathscr{A}^\circ$ are exactly the variables assigned to the free (number and set) variables in $\mathscr{A}$ as described above. Finally, for any $\mathcal{L}_2$ formula $\mathscr{A}[\vec{x}, \vec{X}]$ with free variables among $\vec{x} = x_1, \dots, x_m$,

$\vec{X} = X_1, \ldots, X_m$ we put

$$|\mathscr{A}|_{\mathrm{s}} :\equiv \bigwedge_{i=1}^{m}(x_i^\circ \in \mathrm{N} \wedge X_i^\circ \subseteq \mathrm{N}) \to \mathscr{A}[\vec{x}, \vec{X}]^\circ.$$

The idea of the above translation is that number variables are interpreted as elements of N, and set variables as subsets of N. Atomic formulas are taken care of by involving $\mathrm{val}_t[x]$. Observe that $\mathscr{A}^\circ$ is Δ_0 in case $\mathscr{A}$ is arithmetical. Moreover, Σ_1^1 formulas are carried over to Σ_1 formulas, and Π_1^1 formulas to Π_1 formulas. The following observation will be the basis for further considerations.

Theorem 4.4. *The following hold for any $\mathcal{L}_2$ formula $\mathscr{A}$.*

(a) $\mathsf{ACA}_0 \vdash \mathscr{A}$ *implies* $\mathsf{BS}^0 \vdash |\mathscr{A}|_{\mathrm{s}}$,

(b) $\mathsf{ACA} \vdash \mathscr{A}$ *implies* $\mathsf{BS}^1 \vdash |\mathscr{A}|_{\mathrm{s}}$.

Proof. For (a) it suffices to check that the claim holds for all axioms of ACA_0. For any number-theoretic axiom $\mathscr{A}(\vec{x})$ the claim holds by definition of BS^0 and $\mathscr{A}(\vec{x})^\circ$. The induction axiom is taken care of by (Δ_0-$\mathrm{I_N}$). Similarly, every instance of arithmetical comprehension carries over to an instance of Δ_0 separation, which is available by Theorem 4.3. This shows (a). Since induction for $\mathcal{L}_2$ formulas carries over to ($\mathcal{L}_{\mathrm{s}}$-$\mathrm{I_N}$), we also get (b). □

Note that the above transformation basically embeds $\mathcal{L}_2$ into $\mathcal{L}_{\mathrm{s}}$. Thus, we can view $\mathcal{L}_2$ as a sublanguage of $\mathcal{L}_{\mathrm{s}}$. In this view, Theorem 4.4 states that

$$\mathsf{ACA}_0 \subseteq \mathsf{BS}^0 \quad \text{and} \quad \mathsf{ACA} \subseteq \mathsf{BS}^1.$$

4.3. Objects as trees

As was shown in the previous section, $\mathcal{L}_2$ can be thought of as a sublanguage of $\mathcal{L}_{\mathrm{s}}$. Moreover, we can interpret ACA_0 and ACA in, respectively, BS^0 and BS^1. Aiming for a reversal, we represent objects of $\mathcal{L}_{\mathrm{s}}$ in our base theory ACA_0 with so-called representation trees. In order to identify representation trees coding the same object, we will make use of isomorphisms. However, the existence of these is not provable ad hoc in ACA_0. To overcome this we introduce the formal system ACA_0^+ which extends ACA_0

by allowing arithmetical comprehension to be iterated a finite amount of times. All concepts involved are discussed next.

Definition 4.5. In ACA_0, the notions of n-tree and representation tree are introduced as follows.

(1) T is a n-tree, written $\mathrm{Tree}(n, T)$, iff T is a non-empty tree satisfying
- $n = \max\{\mathrm{lh}(\sigma) : \sigma \in T\}$,
- $(\forall \sigma, x, y)(\sigma * \langle 2x+1 \rangle \in T \wedge y \neq 2x+1 \to \sigma * \langle y \rangle \notin T)$,
- $(\forall \sigma, x)(\sigma * \langle x \rangle \in T \to (\forall i < \mathrm{lh}(\sigma))(\sigma)_i$ is even).

(2) T is a representation tree, in symbols $\mathrm{Rep}(T)$, if T is a n-tree for some natural number n, i.e.,

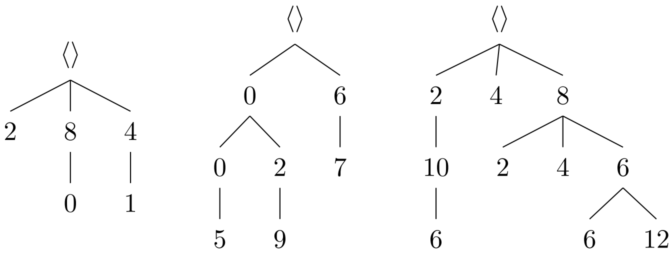
$$\mathrm{Rep}(T) :\equiv \exists n \mathrm{Tree}(n, T).$$

(3) Given a representation tree T and $\sigma \in T$, put $T^\sigma := \{\tau : \sigma * \tau \in T\}$. Moreover, σ is called end node (of T) if $T^\sigma = \{\langle\rangle\}$.

(4) We say that T represents (or codes) a natural number if T is of the form $\{\langle\rangle, \langle 2n+1 \rangle\}$ for some n. Otherwise, T represents a set.

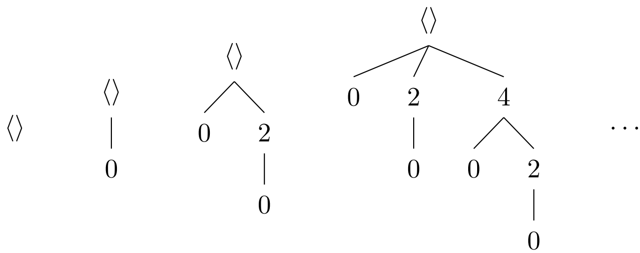
Note that $\mathrm{Rep}(T)$ implies $\mathrm{Rep}(T^\sigma)$. The basic idea is that the natural number n is represented by the 1-tree $\{\langle\rangle, \langle 2n+1 \rangle\}$. In general, a representation tree T stands for the set consisting of all sets represented by the immediate subtrees of T of the form $T^{\langle 2x \rangle}$ with $\langle 2x \rangle \in T$. Working in ACA_0, the assumption $(\forall \sigma \in T)(\mathrm{lh}(\sigma) \leq n)$ allows us to use arithmetical induction to prove basic properties of n-trees.

Example 4.6. We list some representation trees and corresponding sets.

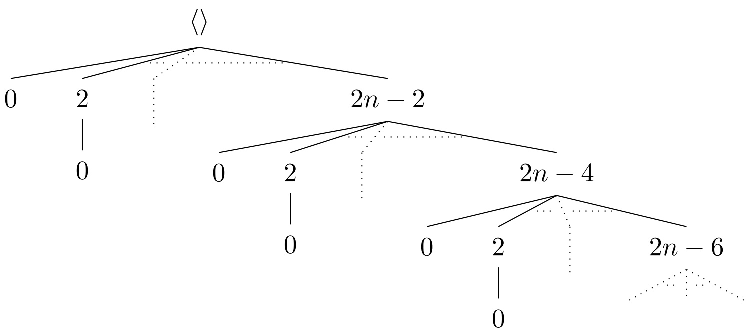


$\{\emptyset, \{\emptyset\}, 0\}$ $\{\{2, 4\}, 3\}$ $\{\{\{\emptyset\}\}, \emptyset, \{\emptyset, \{\emptyset\}\}\}$

Finite ordinals $0, 1, 2, 3, \ldots$ can be represented as follows:



In general, the ordinal number corresponding to $n \in \mathrm{N}$ can be represented by the following representation tree:



More formally, the above tree consists of all $\sigma \in \mathrm{Seq}$ with $\mathrm{lh}(\sigma) \leq n$ such that

$$(\forall i < \mathrm{lh}(\sigma))((\sigma)_i \text{ is even} \wedge (\sigma)_i > (\sigma)_{i+1}) \wedge (\sigma)_0 \leq 2n - 2.$$

Note that it is not possible to represent the first limit ordinal ω, as the corresponding representation tree would contain paths of arbitrary length.

Following [Sim09], we introduce a notion of isomorphism that allows to define object equality and elementhood on the level of representation trees. The main idea is to identify all elements in a representation tree which code the same object. The following definitions clarify this.

Definition 4.7. Given a representation tree T, and a set X, we write $\mathrm{Iso}(X, T)$ to state that $X \subseteq T \times T$ and for all $\sigma, \tau \in T$, $\langle \sigma, \tau \rangle \in X$ iff each of the following properties is fulfilled:

(1) $\forall x(\sigma * \langle 2x \rangle \in T \to \exists y(\langle \sigma * \langle 2x \rangle, \tau * \langle 2y \rangle \rangle \in X))$,

(2) $\forall y(\tau * \langle 2y \rangle \in T \to \exists x(\langle \sigma * \langle 2x \rangle, \tau * \langle 2y \rangle \rangle \in X))$,

(3) $\forall z(\sigma * \langle 2z + 1 \rangle \in T \to \langle \sigma * \langle 2z + 1 \rangle, \tau * \langle 2z + 1 \rangle \rangle \in X)$,

(4) $\forall z(\tau * \langle 2z + 1 \rangle \in T \to \langle \sigma * \langle 2z + 1 \rangle, \tau * \langle 2z + 1 \rangle \rangle \in X)$.

Moreover, we let $\mathrm{C}_{\mathrm{Iso}}(\langle \sigma, \tau \rangle, X, T)$ denote the conjunction of (1) to (4) w.r.t. indicated variables. This formula is introduced to facilitate proving basic properties of representation trees.

Definition 4.8. In ACA_0, we set for all representation trees S and T:

(a) $S \oplus T := \{\langle\rangle\} \cup \{\langle 0 \rangle * \sigma : \sigma \in S\} \cup \{\langle 2 \rangle * \tau : \tau \in T\}$,

(b) $S =^\star T :\equiv \exists X(\mathrm{Iso}(X, S \oplus T) \wedge \langle \langle 0 \rangle, \langle 2 \rangle \rangle \in X)$,

(c) $S \in^\star T :\equiv \exists X(\mathrm{Iso}(X, S \oplus T) \wedge \exists x(\langle \langle 0 \rangle, \langle 2, 2x \rangle \rangle \in X))$.

It is obvious that representation trees are closed under $\oplus$. It remains to check that $=^\star$ and $\in^\star$ have the desired set-theoretic properties within a suitable formal system. As mentioned above, we will introduce ACA_0^+ for that purpose. However, many useful properties can already be proven in ACA_0, which is what we are doing next. As a first observation, we remark that isomorphisms are well-behaved w.r.t. representation trees coding natural numbers. The following examples illustrates this.

Example 4.9. Working in ACA_0, let $\mathcal{R}^n = \{\langle\rangle, \langle 2n+1\rangle\}$ be the representation tree coding the natural number $n \in \mathrm{N}$. Put

$$\mathcal{I}^{m,n} := \left\{\langle\sigma,\tau\rangle \in (\mathcal{R}^m \oplus \mathcal{R}^n)^2 : \mathrm{lh}(\sigma) = \mathrm{lh}(\tau)\right\}.$$

One can show that for $m, n \in \mathrm{Nat}$:

- $\mathcal{R}^m =^\star \mathcal{R}^n \leftrightarrow \mathrm{Iso}(\mathcal{I}^{m,n}, \mathcal{R}^m \oplus \mathcal{R}^n) \leftrightarrow m = n$,
- For all representation trees R and $n \in \mathrm{N}$: $R \notin^\star \mathcal{R}^n$.

In the following, we shall establish further basic properties of representation trees and isomorphisms within ACA_0. We start by showing that isomorphisms are unique and form an equivalence relation.

Lemma 4.10. *The following is provable in* ACA_0*. Consider a representation tree T and sets X, Y satisfying* $\mathrm{Iso}(X,T)$ *and* $\mathrm{Iso}(Y,T)$*. Then $X = Y$. Moreover, X is an equivalence relation on T.*

Proof. Working in ACA_0, suppose $\mathrm{Rep}(T)$, $\mathrm{Iso}(X,T)$ and $\mathrm{Iso}(Y,T)$. By Definition 4.5, let n such that $(\forall\sigma \in T)(\mathrm{lh}(\sigma) \leq n)$. First, we show that $X = Y$. To do so, define the arithmetical formula

$$\mathscr{A}(k) :\equiv \forall\sigma,\tau(k = n \dot{-} \mathrm{lh}(\sigma) \wedge \langle\sigma,\tau\rangle \in X \to \langle\sigma,\tau\rangle \in Y),$$

with $\dot{-}$ denoting the usual truncated subtraction on N. By arithmetical induction we will infer $\forall k \mathscr{A}(k)$, which immediately implies $X \subseteq Y$. Thus, by symmetry, we have $X = Y$. The details of the induction are given next. Let $\langle\sigma,\tau\rangle \in X$. If $k = 0$, we must have $\mathrm{lh}(\sigma) = n$. Thus, σ is an end node. Since $\mathrm{Iso}(X,T)$ and $\langle\sigma,\tau\rangle \in X$, τ must be an end node as well. Hence, $\langle\sigma,\tau\rangle \in Y$ by definition of Y. For the induction step suppose $\mathscr{A}(k)$. To show $\langle\sigma,\tau\rangle \in Y$ where $\mathrm{lh}(\sigma) = n \dot{-} (k+1)$, we check that $\mathrm{C}_{\mathrm{Iso}}(\langle\sigma,\tau\rangle, Y, T)$ holds, cf. Definition 4.7. We only consider one case, the others can be treated analogously. Let x such that $\sigma * \langle 2x\rangle \in T$. Since $\langle\sigma,\tau\rangle \in X$, there exists y such that $\langle\sigma*\langle 2x\rangle, \tau*\langle 2y\rangle\rangle \in X$. Since $\mathrm{lh}(\sigma*\langle x\rangle) = n \dot{-} k$ we obtain by the induction hypothesis that $\langle\sigma*\langle 2x\rangle, \tau*\langle 2y\rangle\rangle \in Y$. This finishes the induction.

We proceed with showing that X constitutes an equivalence relation on T. For reflexivity we show by arithmetical induction that

$$\forall\sigma(k = n \dot{-} \mathrm{lh}(\sigma) \to \langle\sigma,\sigma\rangle \in X)$$

for all k. The assertion holds for $k = 0$ since isomorphisms identify arbitrary end nodes. The induction step follows immediately by verifying $\langle\sigma,\sigma\rangle \in X$ via checking $\mathrm{C}_{\mathrm{Iso}}(\langle\sigma,\sigma\rangle, X, T)$ with the help of the induction hypothesis.

For symmetry it suffices to show for all k by induction that

$$\forall\sigma,\tau(k = n \dot{-} \mathrm{lh}(\sigma) \wedge \langle\sigma,\tau\rangle \in X \to \langle\tau,\sigma\rangle \in X).$$

Recall that $\langle\sigma,\tau\rangle \in X$ implies that τ is an end node, whenever σ is one. From this the case $k = 0$ is immediate. For the induction step we use $\mathrm{C}_{\mathrm{Iso}}(\langle\sigma,\tau\rangle, X, T)$ and the induction hypothesis to infer $\mathrm{C}_{\mathrm{Iso}}(\langle\tau,\sigma\rangle, X, T)$.

For transitivity we show by induction that for all k,

$$\forall\sigma,\rho,\tau(k = n \dot{-} \mathrm{lh}(\sigma) \wedge \langle\sigma,\rho\rangle \in X \wedge \langle\rho,\tau\rangle \in X \to \langle\sigma,\tau\rangle \in X).$$

Arguing as before, the base case is immediate. For the induction step we show $\mathrm{C}_{\mathrm{Iso}}(\langle\sigma,\tau\rangle, X, T)$ using $\mathrm{C}_{\mathrm{Iso}}(\langle\sigma,\rho\rangle, X, T)$, $\mathrm{C}_{\mathrm{Iso}}(\langle\rho,\tau\rangle, X, T)$ and the induction hypothesis. This concludes the proof. $\square$

Next, we establish that isomorphisms are closed under certain operations in ACA_0. The necessary definitions are given first.

Definition 4.11. In ACA_0, let S, T be representation trees, and X, Y sets satisfying $\mathrm{Iso}(X, T)$ and $\mathrm{Iso}(Y, S \oplus T)$. Let $\sigma, \tau \in S$. For $\alpha \in T^\sigma \oplus T^\tau$ we put:

$$\alpha^+ := \begin{cases} \langle\rangle & \text{if } \alpha = \langle\rangle, \\ \sigma * \gamma & \text{if } \alpha = \langle 0\rangle * \gamma \text{ for some } \gamma \in T^\sigma, \\ \tau * \gamma & \text{if } \alpha = \langle 2\rangle * \gamma \text{ for some } \gamma \in T^\tau. \end{cases}$$

Then we set $X^{\langle\sigma,\tau\rangle} := \big\{\langle\alpha,\beta\rangle \in (T^\sigma \oplus T^\tau)^2 : \langle\alpha^+,\beta^+\rangle \in X\big\}$.

Moreover, given $\alpha \in T \oplus S$ we let

$$\alpha^{-1} := \begin{cases} \langle\rangle & \text{if } \alpha = \langle\rangle, \\ \langle 0\rangle * \gamma & \text{if } \alpha = \langle 2\rangle * \gamma \text{ for some } \gamma \in S, \\ \langle 2\rangle * \gamma & \text{if } \alpha = \langle 0\rangle * \gamma \text{ for some } \gamma \in T. \end{cases}$$

Then we put $Y^{-1} := \big\{\langle\alpha,\beta\rangle \in (T \oplus S)^2 : \langle\alpha^{-1},\beta^{-1}\rangle \in Y\big\}$.

Lemma 4.12. *Working in* ACA_0*, let* S, T *be representation trees, and* X, Y *sets satisfying* $\mathrm{Iso}(X, T)$ *and* $\mathrm{Iso}(Y, S \oplus T)$*. The following assertions hold:*

(a) For all $\sigma, \tau \in T$*:* $\mathrm{Iso}(X^{\langle \sigma, \tau \rangle}, T^\sigma \oplus T^\tau)$*,*

(b) $\mathrm{Iso}(Y^{-1}, T \oplus S)$*.*

In addition, $X^{\langle \sigma, \tau \rangle}, Y^{-1}$ *are uniquely determined.*

Proof. All assertions can be directly proven by looking at Definition 4.7 and unwrapping Definition 4.11. It is also clear that the given sets exist properly using arithmetical comprehension with, respectively, X or Y as parameter. Uniqueness is immediate by Lemma 4.10. □

We continue with discussing a fundamental lemma which provides a link between isomorphisms and the relations $=^\star$ and $\in^\star$ in ACA_0. After having established the existence of isomorphisms in ACA_0^+, this will facilitate proving the set-theoretic properties of representation trees we require.

Lemma 4.13. *The following is provable in* ACA_0*. Let* T *be a representation tree, and* X *a set such that* $\mathrm{Iso}(X, T)$*. For* $\sigma, \tau \in T$ *we then have:*

(a) $T^\sigma =^\star T^\tau \leftrightarrow \langle \sigma, \tau \rangle \in X$*,*

(b) $T^\sigma \in^\star T^\tau \leftrightarrow \exists x(\langle \sigma, \tau * \langle 2x \rangle \rangle \in X)$*.*

Proof. Working in ACA_0, consider $\sigma, \tau \in T$. By Lemma 4.12 we have $\mathrm{Iso}(X^{\langle \sigma, \tau \rangle}, T^\sigma \oplus T^\tau)$. Moreover, $X^{\langle \sigma, \tau \rangle}$ is uniquely determined and defined in terms of X. Looking at Definition 4.8, it suffices to show that

$$\langle \langle 0 \rangle, \langle 2 \rangle \rangle \in X^{\langle \sigma, \tau \rangle} \leftrightarrow \langle \sigma, \tau \rangle \in X.$$

By definition of $X^{\langle \sigma, \tau \rangle}$, this follows immediately, proving (a).
(b): We argue similarly as for (a). It suffices to show that for all x,

$$\langle \langle 0 \rangle, \langle 2, 2x \rangle \rangle \in X^{\langle \sigma, \tau \rangle} \leftrightarrow \langle \sigma, \tau * \langle 2x \rangle \rangle \in X.$$

By construction of $X^{\langle \sigma, \tau \rangle}$, this is easily verified, showing (b). □

Before we can introduce the system ACA_0^+ and show that it establishes the existence of isomorphisms, we need an additional auxiliary notion.

Definition 4.14. The following definition is made in ACA_0. Let T be a representation tree and $\tau \in T$. Then we set

$$\mathrm{lh}_T^{-1}(\tau) := \max \{\mathrm{lh}(\sigma) : \sigma \in T^\tau\}.$$

Remark 4.15. For any representation tree T and $\sigma * \langle x \rangle \in T$ we have in ACA_0 that

$$\mathrm{lh}_T^{-1}(\sigma * \langle x \rangle) < \mathrm{lh}_T^{-1}(\sigma).$$

Definition 4.16. Let $\mathscr{A}(n, j, U)$ be an arithmetical formula of $\mathcal{L}_2$ with potentially other free variables than the ones displayed.

(a) We define $\mathcal{H}^{\cup}_{\mathscr{A}}(k, Y)$ to be the formula asserting that

$$Y = \left\{ \langle n, j \rangle : j \leq k \wedge \mathscr{A}(n, j, \textstyle\bigcup_{i<j} (Y)_i) \right\}.$$

(b) ACA_0^+ is the formal system in $\mathcal{L}_2$ consisting of ACA_0 plus all formulas of the form

$$\forall k \exists Y \mathcal{H}^{\cup}_{\mathscr{A}}(k, Y),$$

where the formula $\mathscr{A}(n, j, U)$ is given as above.

Remark 4.17. Note that ACA_0^+ is contained in ACA_0 plus induction for Σ^1_1 formulas or in ATR_0. Thus, we have

$$\mathsf{ACA}_0^+ \subseteq \mathsf{ACA} \quad \text{and} \quad \mathsf{ACA}_0^+ \subseteq \mathsf{ATR}_0.$$

Theorem 4.18. *The following is provable in ACA_0^+ for any set T:*

$$\mathrm{Rep}(T) \rightarrow \exists Z \mathrm{Iso}(Z, T).$$

Proof. We work in ACA_0^+. To prove the assertion we define the arithmetical formula

$$\mathscr{A}(n, j, Y) :\equiv (\exists \sigma, \tau \in T)(n = \langle \sigma, \tau \rangle \wedge \mathrm{lh}_T^{-1}(\sigma) = j \wedge \mathrm{C}_{\mathrm{Iso}}(n, Y, T)).$$

Suppose T is a representation tree and let k such that $\mathrm{Tree}(k, T)$. By definition of ACA_0^+, there exists Y such that

$$\mathcal{H}^{\cup}_{\mathscr{A}}(k, Y).$$

We claim that $Z := \bigcup_{j=0}^{k}(Y)_j$ satisfies $\mathrm{Iso}(Z,T)$. By construction we have $(Y)_j \subseteq T \times T$ for all $j \leq k$. Thus, also $Z \subseteq T \times T$. It remains to show that for $\sigma, \tau \in T$:

$$\langle \sigma, \tau \rangle \in Z \leftrightarrow \mathrm{C}_{\mathrm{Iso}}(\langle \sigma, \tau \rangle, Z, T).$$

Before proving the above we note that by Remark 4.15 we have

$$j = \mathrm{lh}_T^{-1}(\sigma) \wedge \mathrm{C}_{\mathrm{Iso}}(\langle \sigma, \tau \rangle, Z, T) \to \mathrm{C}_{\mathrm{Iso}}(\langle \sigma, \tau \rangle, \textstyle\bigcup_{i<j}(Y)_i, T). \tag{4.1}$$

Now, assume $\langle \sigma, \tau \rangle \in Z$. Then $\langle \sigma, \tau \rangle \in (Y)_j$ for some $j \leq k$. $\mathcal{H}^{\cup}_{\mathscr{A}}(k, Y)$ implies that $\mathrm{C}_{\mathrm{Iso}}(\langle \sigma, \tau \rangle, \bigcup_{i<j}(Y)_i, T)$. It follows that $\mathrm{C}_{\mathrm{Iso}}(\langle \sigma, \tau \rangle, Z, T)$ since $\bigcup_{i<j}(Y)_i \subseteq Z$. Conversely, suppose $\mathrm{C}_{\mathrm{Iso}}(\langle \sigma, \tau \rangle, Z, T)$. By (4.1) we get

$$\mathrm{C}_{\mathrm{Iso}}(\langle \sigma, \tau \rangle, \textstyle\bigcup_{i<j}(Y)_i, T),$$

which amounts to $\langle \sigma, \tau \rangle \in (Y)_j \subseteq Z$. This finishes the proof. □

Finally, we have all the ingredients to prove the required set-theoretic properties of representation trees within ACA_0^+. In particular, we can now show that $=^\star$ forms an equivalence relation and fulfils extensionality.

Theorem 4.19. *Working in ACA_0^+, let R, S, T be representation trees. Then we can show the following.*

(a) $=^\star$ is an equivalence relation on the class of representation trees.

(b) If S and T are coding sets, then

$$S =^\star T \leftrightarrow \forall W (W \in^\star S \leftrightarrow W \in^\star T).$$

(c) $R =^\star S \wedge R \in^\star T \to S \in^\star T$.

(d) $R \in^\star S \wedge S =^\star T \to R \in^\star T$.

(e) $S \in^\star T \leftrightarrow \exists x (S =^\star T^{\langle 2x \rangle})$.

Proof. In the following, we tacitly use Lemma 4.10, Lemma 4.12 and Theorem 4.18.
(a): Our goal is to show that $=^\star$ is reflexive, symmetric and transitive. For reflexivity we show $T =^\star T$. Consider X such that $\mathrm{Iso}(X,T)$. Recall that $\mathrm{Iso}(X^{\langle\langle\rangle,\langle\rangle\rangle}, T \times T)$ and $\langle\langle\rangle,\langle\rangle\rangle \in X$. By construction of $X^{\langle\langle\rangle,\langle\rangle\rangle}$ we

therefore get $\langle\langle 0\rangle, \langle 2\rangle\rangle \in X^{\langle\langle\rangle,\langle\rangle\rangle}$, verifying $T =^\star T$. For symmetry we assume $S =^\star T$. Let Y such that $\mathrm{Iso}(Y, S \oplus T)$. By assumption we have $\langle\langle 2\rangle, \langle 0\rangle\rangle \in Y$. Recall that $\mathrm{Iso}(Y^{-1}, T \oplus S)$. It follows that $\langle\langle 0\rangle, \langle 2\rangle\rangle \in Y^{-1}$, verifying $T =^\star S$. For transitivity, assume $R =^\star S$ and $S =^\star T$. Define the representation tree U as follows

$$P := \{\langle\rangle\} \cup \{\langle 0\rangle * \rho : \rho \in R\} \cup \{\langle 2\rangle * \sigma : \sigma \in S\} \cup \{\langle 4\rangle * \tau : \tau \in T\}.$$

Let Y such that $\mathrm{Iso}(Y, P)$. Note that

$$P^{\langle 0\rangle} = R, \quad P^{\langle 2\rangle} = S, \quad P^{\langle 4\rangle} = T.$$

Since $R =^\star S$ and $S =^\star T$, we obtain by Lemma 4.13 that

$$\langle\langle 0\rangle, \langle 2\rangle\rangle, \langle\langle 2\rangle, \langle 4\rangle\rangle \in Y.$$

It follows that $\langle\langle 0\rangle, \langle 4\rangle\rangle \in Y$, which by Lemma 4.13 amounts to $R =^\star T$, as desired.

(b): First, suppose $S =^\star T$ and let W such that $W \in^\star S$. Let P be the following representation tree,

$$P := \{\langle\rangle\} \cup \{\langle 0\rangle * \rho : \rho \in W\} \cup \{\langle 2\rangle * \sigma : \sigma \in S\} \cup \{\langle 4\rangle * \tau : \tau \in T\}.$$

Note that

$$P^{\langle 0\rangle} = W, \quad P^{\langle 2\rangle} = S, \quad P^{\langle 4\rangle} = T.$$

Let Y such that $\mathrm{Iso}(Y, P)$. By assumption, Lemma 4.13 implies that $\langle\langle 2\rangle, \langle 4\rangle\rangle \in Y$ and $\langle\langle 0\rangle, \langle 2, 2x\rangle\rangle \in Y$ for some x. Let y such that

$$\langle\langle 2, 2x\rangle, \langle 4, 2y\rangle\rangle \in Y.$$

It follows that $\langle\langle 0\rangle, \langle 4, 2y\rangle\rangle \in Y$, yielding $W \in^\star S$ by Lemma 4.13. If $W \in^\star T$, we can proceed analogously.

For the converse direction, suppose

$$\forall W (W \in^\star S \leftrightarrow W \in^\star T).$$

Consider Y such that $\mathrm{Iso}(Y, S \oplus T)$. It suffices to show $\langle\langle 0\rangle, \langle 2\rangle\rangle \in Y$. We do so by proving $\mathrm{C}_{\mathrm{Iso}}(\langle\langle 0\rangle, \langle 2\rangle\rangle, Y, S \oplus T)$. Recall that S, T code sets, therefore there is no x such that $\langle 2x + 1\rangle \in S \cup T$. Now, let x such that

$\langle 2x\rangle \in S$. By Lemma 4.13 we have $S^{\langle 2x\rangle} \in^\star S$. By assumption $S^{\langle 2x\rangle} \in^\star T$. Observe that

$$Y^{\langle 0,2x\rangle} = S^{\langle 2x\rangle}, \quad Y^{\langle 2\rangle} = T.$$

Hence, by Lemma 4.13 there exists y such that

$$\langle\langle 0, 2x\rangle, \langle 2, 2y\rangle\rangle \in Y.$$

If y satisfies $\langle 2y\rangle \in T$, we can proceed similarly. This establishes

$$\mathrm{C}_{\mathrm{Iso}}(\langle\langle 0\rangle, \langle 2\rangle\rangle, Y, S \oplus T),$$

concluding the proof of (b).
(c): Assume $R =^\star S$ and $R \in^\star T$. We have to show $S \in^\star T$. Consider

$$P := \{\langle\rangle\} \cup \{\langle 0\rangle * \rho : \rho \in R\} \cup \{\langle 2\rangle * \sigma : \sigma \in S\} \cup \{\langle 4\rangle * \tau : \tau \in T\}$$

and Y with $\mathrm{Iso}(Y, P)$. By assumption and Lemma 4.13 we get $\langle\langle 0\rangle, \langle 2\rangle\rangle \in Y$ and $\langle\langle 0\rangle, \langle 4, 2x\rangle\rangle \in Y$ for some x. It follows that $\langle\langle 2\rangle, \langle 4, 2x\rangle\rangle \in Y$, verifying $S \in^\star T$ by Lemma 4.13.
(d): Assume $S =^\star T$ and $R \in^\star S$. Our goal is to show $R \in^\star T$. Let P, Y be as above in the proof of (c). By assumption and Lemma 4.13 we obtain $\langle\langle 2\rangle, \langle 4\rangle\rangle \in Y$ and $\langle\langle 0\rangle, \langle 2, 2x\rangle\rangle \in Y$ for some x. By definition of Y, there exists y such that $\langle\langle 2, 2x\rangle, \langle 4, 2y\rangle\rangle \in Y$. It follows that $\langle\langle 0\rangle, \langle 4, 2y\rangle\rangle \in Y$, yielding $R \in^\star T$ using Lemma 4.13.
(e): Assume $S \in^\star T$. Let Y such $\mathrm{Iso}(Y, S \oplus T)$. By assumption, there exists x such that $\langle\langle 0\rangle, \langle 2, 2x\rangle\rangle \in Y$. Since $(S \oplus T)^{\langle 0\rangle} = S$ and $(S \oplus T)^{\langle 2,2x\rangle} = T^{\langle 2x\rangle}$, Lemma 4.13 immediately implies $S =^\star T^{\langle 2x\rangle}$. The other direction is immediate by (d) since $T^{\langle 2x\rangle} \in^\star T$ if $T \neq \{\langle\rangle\}$. This is immediate by Lemma 4.13. □

By Theorem 4.19, it follows that $=^\star$ behaves like equality on the class of representation trees and is compatible with $\in^\star$. Moreover, $\in^\star$ is extensional and for every set coded by some representation tree, the elements of that set are exactly the sets coded by the immediate subtrees of the given tree. To conclude this section we prove some additional lemmas that will enable us to properly interpret ($\mathcal{L}_{\mathrm{s}}$-$\mathrm{I}_\in$) in ACA. This will be carried out in the next chapter.

Lemma 4.20. *Working in* ACA_0*, consider a representation tree* T *and set* X *such that* $\mathrm{Iso}(X, T)$*. Then we can prove for* $\sigma, \tau \in T$ *that*

$$\langle \sigma, \tau \rangle \in X \to \mathrm{lh}_T^{-1}(\sigma) = \mathrm{lh}_T^{-1}(\tau).$$

Proof. We work in ACA_0. By definition of T, there exists n such that $n = \max \{\mathrm{lh}(\tau) : \tau \in T\}$. Let $\mathscr{A}(k)$ be the formula

$$\forall \sigma, \tau (k = n \dot{-} \mathrm{lh}(\sigma) \wedge \langle \sigma, \tau \rangle \in X \to \mathrm{lh}_T^{-1}(\sigma) = \mathrm{lh}_T^{-1}(\tau)).$$

By arithmetical induction we show $\mathscr{A}(k)$ for all k. From this the claim follows. Let $\langle \sigma, \tau \rangle \in X$. Suppose $k = 0$. Then $\mathrm{lh}(\sigma) = n$, therefore σ must be an end node of T. Since $\langle \sigma, \tau \rangle \in X$, this is also the case for τ. It follows that $\mathrm{lh}_T^{-1}(\sigma), \mathrm{lh}_T^{-1}(\tau)$ are both 0. For the induction step, suppose $\mathscr{A}(k)$ and $k + 1 = n \dot{-} \mathrm{lh}(\sigma)$. By the induction hypothesis we know that

$$\left\{ \mathrm{lh}_T^{-1}(\sigma * \langle x \rangle) : \sigma * \langle x \rangle \in T \right\} = \left\{ \mathrm{lh}_T^{-1}(\tau * \langle y \rangle) : \tau * \langle y \rangle \in T \right\}.$$

The above implies $\mathrm{lh}_T^{-1}(\sigma) = \mathrm{lh}_T^{-1}(\tau)$, proving the assertion. $\square$

Lemma 4.21. *Working in* ACA_0^+*, let* S, T *be representation trees. Then we have*

$$\mathrm{Tree}(m, S) \wedge \mathrm{Tree}(n, T) \wedge S \in^\star T \to m < n.$$

Proof. Working in ACA_0^+, assume that $\mathrm{Tree}(m, S)$, $\mathrm{Tree}(n, T)$ and $S \in^\star T$. We have to show $m < n$. Let Y such that $\mathrm{Iso}(Y, S \oplus T)$. Since $S \in^\star T$ we get

$$\langle \langle 0 \rangle, \langle 2, 2x \rangle \rangle \in Y$$

for some x. Using Lemma 4.20 and Remark 4.15 we obtain

$$m = \mathrm{lh}_{S \oplus T}^{-1}(\langle 0 \rangle) = \mathrm{lh}_{S \oplus T}^{-1}(\langle 2, 2x \rangle) < \mathrm{lh}_{S \oplus T}^{-1}(\langle 2 \rangle) = n.$$

This finishes the proof. $\square$

5. Interpreting BS, BS^1 and BS^0 in Second Order Arithmetic

The goal of this chapter is to embed $\mathsf{BS}^0, \mathsf{BS}^1$ and BS into subsystems of second order arithmetic. A central role will be played by the theory ACA_0^+. It takes care of the number-theoretic axioms, the ontological axioms, (Pair), and the Rudimentary Closure axioms. For dealing with (Δ_0-$\mathrm{I_N}$) we add (Σ^1_1-AC). Finally, full induction on the natural numbers and full $\in$-induction can be handled in ACA. To achieve all of this, we translate from $\mathcal{L}_\mathrm{s}$ to $\mathcal{L}_2$ using representation trees. We will then validate all axioms, or rather their translations, on the class of representation trees. Please note that $\mathsf{ACA}_0^+ + (\Sigma^1_1\text{-AC})$ is weaker than ACA w.r.t. proof-theretic strength, cf. [Buc05].

5.1. Translating from $\mathcal{L}_\mathrm{s}$ to $\mathcal{L}_2$

We will employ representation trees to translate $\mathcal{L}_\mathrm{s}$ formulas into $\mathcal{L}_2$ formulas. Recall that we use $\chi_0, \chi_1, \dots$ to denote the object variables of $\mathcal{L}_\mathrm{s}$, and $\mathcal{V}_0, \mathcal{V}_1, \dots$ to denote the set variables of $\mathcal{L}_2$. The following definition makes this transformation precise.

Definition 5.1. For every term t of $\mathcal{L}_\mathrm{s}$, we associate a term $t^\star$ of $\mathcal{L}_2$ as follows.

- $\chi_i^\star := \mathcal{V}_i$ for $i \in \mathbb{N}$,
- $\underline{n}^\star := \big\{\langle\rangle, \langle\overline{2n+1}\rangle\big\}$ for all (standard) natural numbers $n \in \mathbb{N}$,
- $\mathrm{N}^\star := \{\langle\rangle\} \cup \{\sigma : \exists x(\sigma = \langle 2x\rangle \vee \sigma = \langle 2x, 2x+1\rangle)\}$.

Technically, the $\underline{n}^\star$'s and $\mathrm{N}^\star$ are class terms. The literals of $\mathcal{L}_\mathrm{s}$ are transformed as follows.

- $(s \in t)^\star :\equiv s^\star \in^\star t^\star$,

- $(s =_{\mathrm{N}} t) :\equiv \exists x(s^\star = t^\star = \{\langle\rangle, \langle 2x+1\rangle\})$,
- $\mathrm{S}(t)^\star :\equiv \forall x(t^\star \neq \{\langle\rangle, \langle 2x+1\rangle\})$,
- $Q(t)^\star :\equiv \exists x(t^\star = \{\langle\rangle, \langle 2x+1\rangle\} \wedge Q(x))$.
- For any n-ary p.r. relation symbol R,

$$R(t_1, \ldots, t_n)^\star :\equiv (\exists x_1, \ldots, x_n)$$
$$(\textstyle\bigwedge_{i=1}^{n} t_i^\star = \{\langle\rangle, \langle 2x_i+1\rangle\} \wedge R(x_1, \ldots, x_n))\,.$$

Moreover, the propositional connectives commute with $\star$, and for the quantifiers we set

- $(\exists \chi_i \mathscr{A})^\star :\equiv \exists \chi_i^\star(\mathrm{Rep}(\chi_i^\star) \wedge \mathscr{A}^\star)$,
- $(\forall \chi_i \mathscr{A})^\star :\equiv \forall \chi_i^\star(\mathrm{Rep}(\chi_i^\star) \to \mathscr{A}^\star)$.

Given a list $\vec{U} = U_1, \ldots, U_n$ of set variables of $\mathcal{L}_2$, we use $\mathrm{Rep}(\vec{U})$ as an abbreviation for

$$\mathrm{Rep}(U_1) \wedge \ldots \wedge \mathrm{Rep}(U_n).$$

Finally, for any formula $\mathscr{A}[\vec{u}]$ of $\mathcal{L}_\mathrm{s}$ with at most the object variables $\vec{u} = u_1, \ldots, u_n$ free, we associate the $\mathcal{L}_2$ formula

$$|\mathscr{A}|_2 :\equiv \mathrm{Rep}(\vec{u}^\star) \to \mathscr{A}[\vec{u}]^\star,$$

where $\vec{u}^\star = u_1^\star, \ldots u_n^\star$.

In the following section we will employ the above transformation to validate all set-theoretic axioms on the class of representation trees. All axioms except the induction principles (Δ_0-I_N), ($\mathcal{L}_\mathrm{s}$-I_N) and ($\mathcal{L}_\mathrm{s}$-$\mathrm{I}_\in$) can be taken care of in ACA_0^+. To take care of these, we introduce the system Σ_1^1-AC_0^+.

Definition 5.2. Σ_1^1-AC_0^+ is the formal system in $\mathcal{L}_2$ consisting of ACA_0^+ together with the axiom schema (Σ_1^1-AC).

Before going into the details of validating all axioms, we discuss how the above translation affects the complexity of formulas. Before we can present the lemma dealing with that question, we recall the definition of Σ and Π formulas of $\mathcal{L}_\mathrm{s}$. Moreover, we need to show that translations of $\mathcal{L}_\mathrm{s}$ formulas are closed under substitution of representation trees.

Definition 5.3. The class of Σ formulas is the smallest class of $\mathcal{L}_\mathrm{s}$ formulas that contains the Δ_0 formulas, and is closed under the connectives $\wedge$ and $\vee$, as well as existential quantifiers. The class of Π formulas is defined exactly as the Σ formulas, but instead of closure under existential quantifiers, we require closure under universal quantifiers.

Lemma 5.4. *The following is provable in* ACA_0^+. *Consider the* $\mathcal{L}_\mathrm{s}$ *formula* $\mathscr{A}(u_1, \ldots, u_n)$, *and a list of object variables* $v_1, \ldots, v_n$. *Then we have*

$$\bigwedge_{i=1}^{n}(u_i^\star =^\star v_i^\star) \to (|\mathscr{A}|_2(u_1^\star, \ldots, u_n^\star) \to |\mathscr{A}|_2(v_1^\star, \ldots, v_n^\star)).$$

Proof. Working in ACA_0^+, the assertion can be proven directly by induction on the build-up of $\mathscr{A}(\vec{u})$. For the atomic cases we use Theorem 4.19 and the fact that $\mathrm{N}^\star$ and $\underline{n}^\star$ properly define representation trees, where $n \in \mathbb{N}$. If $\mathscr{A}(\vec{u})$ is built up using propositional connectives or object quantifiers, the assertion follows by the induction hypothesis. □

Lemma 5.5. *The following is provable in* ACA_0^+.

(a) For any Δ_0 *formula* $\mathscr{A}$, $|\mathscr{A}|_2$ *is provably equivalent to a* Σ^1 *and a* Π^1 *formula, both having the same free variables as* $|\mathscr{A}|_2$.

(b) For any Σ *formula* $\mathscr{A}$, $|\mathscr{A}|_2$ *is provably equivalent to a* Σ^1 *formula with the same free variables as* $|\mathscr{A}|_2$.

(c) For any Π *formula* $\mathscr{A}$, $|\mathscr{A}|_2$ *is provably equivalent to a* Π^1 *formula with the same free variables as* $|\mathscr{A}|_2$.

Proof. (a): Let $\mathscr{A}[\vec{u}]$ be a Δ_0 formula of $\mathcal{L}_\mathrm{s}$ with free object variables among $\vec{u} = u_1, \ldots, u_n$. We work in ACA_0^+ and proceed by induction on the build-up of $\mathscr{A}[\vec{u}]$. In the following we will always assume that $\mathrm{Rep}(u_1^\star, \ldots, u_n^\star)$. It suffices to show the claim for $\mathscr{A}^\star(u_1^\star, \ldots, u_n^\star)$. Suppose that $\mathscr{A}[\vec{u}]$ is of the form $u_i \in u_j$. We show that $u_i^\star \in^\star u_j^\star$ is Δ_1^1. By definition it is equivalent to a Σ_1^1 formula. By Lemma 4.10 and Theorem 4.18 it is equivalent to the Π_1^1 formula

$$\forall X(\mathrm{Iso}(X, u_i^\star \oplus u_j^\star) \to \forall x(\langle\langle 0\rangle, \langle 2, x\rangle\rangle \in X)).$$

For the remaining atomic cases the claim is obvious. Recall that the defining formulas of the representation trees $\mathrm{N}^\star$ and $\underline{n}^\star$ for $n \in \mathbb{N}$ are arithmetical.

If $\mathscr{A}[\vec{u}]$ is a negative literal or of the form $\mathscr{B}_1[\vec{u}] \diamond \mathscr{B}_2[\vec{u}]$ with $\diamond \in \{\wedge, \vee\}$, the claim follows by the induction hypothesis. Finally, suppose $\mathscr{A}[\vec{u}]$ is of the form $(\mathrm{Q}x \in u_i)\mathscr{B}[x, \vec{u}]$ with $\mathrm{Q} \in \{\forall, \exists\}$. It suffices to observe that from Theorem 4.19(e) and Lemma 5.4 we can infer,

$$\forall X(\mathrm{Rep}(X) \wedge X \in^\star u_i^\star \rightarrow \mathscr{B}^\star[X, \vec{u}^\star]) \leftrightarrow \forall x \mathscr{B}^\star[u_i^{\star\langle 2x \rangle}, \vec{u}^\star],$$
$$\exists X(\mathrm{Rep}(X) \wedge X \in^\star u_i^\star \wedge \mathscr{B}^\star[X, \vec{u}^\star]) \leftrightarrow \exists x \mathscr{B}^\star[u_i^{\star\langle 2x \rangle}, \vec{u}^\star].$$

Note that $\mathrm{Rep}(X)$ is arithmetical. This is enough to establish (a).
(b): Let $\mathscr{A}[\vec{u}]$ be a Σ formula with variables among $u = u_1, \ldots, u_n$. We proceed inductively as for (a). Assume $\mathrm{Rep}(u_1^\star, \ldots, u_n^\star)$. We can restrict to the case that $\mathscr{A}[\vec{u}]$ is of the form $\exists x \mathscr{B}[x, \vec{u}]$. All other cases are analogously to or implied by (a). In particular, $u_i^\star \in^\star u_j^\star$ is Σ^1_1. By the induction hypothesis, the formula

$$\exists X(\mathrm{Rep}(X) \wedge \mathscr{B}^\star[X, u_1^\star, \ldots, u_n^\star])$$

is Σ^1. This suffices to prove (b).
(c): To establish the claim we can proceed analogously as for (b), the only interesting case being when $\mathscr{A}[\vec{u}]$ is of the form $\exists x \mathscr{B}[x, \vec{u}]$. It suffices to note that we can infer inductively that the formula

$$\forall X(\mathrm{Rep}(X) \rightarrow \mathscr{B}^\star[X, u_1^\star, \ldots, u_n^\star])$$

is Π^1. This finishes the proof. $\square$

Corollary 5.6. *The following is provable in* Σ^1_1-AC_0^+. *For any* Δ_0 *formula* $\mathscr{A}$, $|\mathscr{A}|_2$ *is* Δ^1_1, *i.e., it is provably equivalent to a* Σ^1_1 *and a* Π^1_1 *formula, both having the same free variables as* $|\mathscr{A}|_2$.

Proof. We work in Σ^1_1-AC_0^+. By Corollary 2.4 it suffices to show that $|\mathscr{A}|_2$ is provably equivalent to a Σ^1 and a Π^1 formula. This is ensured by Lemma 5.5(a). $\square$

To conclude the section we establish that within ACA_0, any formula $\mathscr{A}$ of $\mathcal{L}_2$ is (in some sense) equivalent to $||\mathscr{A}|_s|_2$. Recall that the involved transformations are defined in, respectively, section 4.2 and Definition 5.1. Intuitively, this means that representation trees faithfully represent natural numbers and subsets thereof in the context of ACA_0. This shall now be checked in more detail.

Definition 5.7. Working in ACA_0, let X be any set. We say that X represents the number z, written $X =_{\mathrm{obj}} z$, if $X = \{\langle\rangle, \langle 2z+1\rangle\}$. Moreover, given a set Z, X represents Z, written $X =_{\mathrm{obj}} Z$, if

$$X = \{\langle\rangle\} \cup \{\langle 2z\rangle : z \in Z\} \cup \{\langle 2z, 2z+1\rangle : z \in Z\}\,.$$

In other words, X represents a number or a set if X is a fixed representation tree coding the respective object.

Proposition 5.8. *The following is provable in* ACA_0*. Let* $t[\vec{x}]$ *be any term of* $\mathcal{L}_2$ *with variables among* $\vec{x} = x_1, \ldots, x_n$*, and* y *an object variable of* $\mathcal{L}_\mathrm{s}$*. Then* $|\mathrm{val}_t[y]|_2$ *is equivalent to the formula*

$$(\exists u_1, \ldots, u_n, v)(y^\star =_{\mathrm{obj}} v \wedge \textstyle\bigwedge_{i=1}^n (x_i^\circ)^\star =_{\mathrm{obj}} u_i \wedge v = t[\vec{u}]).$$

Proof. Working in ACA_0, the claim follows by induction on the build-up of $t[\vec{x}]$. Note that if $X =_{\mathrm{obj}} z$ for some set X and number z, then both X and z are uniquely determined. Suppose $t[\vec{x}]$ is x_i. Then $\mathrm{val}_t[y] \equiv y =_\mathrm{N} x_i^\circ$. Hence, $|\mathrm{val}_t[y]|_2$ is equivalent to the formula

$$\exists u, v(y^\star =_{\mathrm{obj}} v \wedge (x_i^\circ)^\star =_{\mathrm{obj}} u \wedge v = u).$$

From this, the assertion follows easily. Next, suppose that $t[\vec{x}]$ is of the form $f(s_1[\vec{x}], \ldots, s_m[\vec{x}])$ for some m-ary p.r. function symbol f. Then $\mathrm{val}_t[y]$ is of the form

$$(\exists y_1, \ldots, y_m \in \mathrm{N})(\textstyle\bigwedge_{j=1}^m \mathrm{val}_{s_j}[y_j] \wedge \mathrm{R}_f(y_1, \ldots, y_m, y)),$$

with R_f being the p.r. relation symbol for the graph of f. By the induction hypothesis we have that for all $j = 1, \ldots, m$, $|\mathrm{val}_{s_j}[y_j]|_2$ holds iff

$$(\exists u_1, \ldots, u_n, v_j)(y_j^\star =_{\mathrm{obj}} v_j \wedge \textstyle\bigwedge_{i=1}^n (x_i^\circ)^\star =_{\mathrm{obj}} u_i \wedge v_j = s_j[\vec{u}]).$$

The conjunction of all the above formulas with j ranging over $j = 1, \ldots, m$

is equivalent to

$$(\exists \vec{u}, \vec{v})(\bigwedge_{j=1}^{m} y_j^\star =_{\text{obj}} v_j \wedge \bigwedge_{i=1}^{n} (x_i^\circ)^\star =_{\text{obj}} u_i \wedge v_j = s_j[\vec{u}]),$$

where $\vec{u} = u_1, \ldots, u_n$, $\vec{v} = v_1, \ldots, v_m$. By properties of R_f and since $X \in^\star \mathrm{N}^\star \leftrightarrow \exists z(X =_{\text{obj}} z)$ and $\forall z \exists X(X =_{\text{obj}} z)$, it is now straightforward to establish the desired assertion for $|\mathrm{val}_t[y]|_2$. □

Lemma 5.9. *Let $\mathscr{A}$ be a formula of $\mathcal{L}_2$, $\mathscr{B}$ its translation into $\mathcal{L}_{\mathrm{s}}$, and $\mathscr{C}$ the translation of $\mathscr{B}$ back into $\mathcal{L}_2$. Assume further that $\vec{x}, \vec{X}$ lists all free number and set variables of $\mathscr{A}$, $\vec{y}$ is the list of the corresponding object variables of $\mathscr{B}$, and $\vec{Z}$ the set variables of $\mathscr{C}$ that correspond to $\vec{y}$. Observe that $\mathscr{C}$ does not contain free number variables. Then* ACA_0^+ *proves that*

$$\vec{Z} =_{\text{obj}} \vec{x}, \vec{X} \to (\mathscr{A} \leftrightarrow \mathscr{C}).$$

Proof. Working in ACA_0^+, the assertion can be proved by induction on the build-up of $\mathscr{A}[\vec{x}, \vec{X}]$. Note that if X represents an object, i.e., $X =_{\text{obj}} z$ or $X =_{\text{obj}} Z$ for some number z or set Z, then X is uniquely determined up to isomorphism. Moreover, every number and set is represented by some unique set. It is now straightforward to carry out the induction by employing Proposition 5.8, Lemma 5.4 and the following additional properties:

- $X \in^\star \mathrm{N}^\star \leftrightarrow \exists z(X =_{\text{obj}} z)$,
- $Y \subseteq^\star \mathrm{N}^\star \leftrightarrow \exists X, Z(X =^\star Y \wedge X =_{\text{obj}} Z)$,

where $Y \subseteq^\star \mathrm{N}^\star :\equiv (\forall T \in^\star Y^\star)(T \in^\star \mathrm{N}^*)$. We omit the details. □

5.2. Interpreting (RC) and co.

The main goal of this section is to validate the Rudimentary Closure axioms of BS in ACA_0^+ on the class of representation trees. However, we first deal with the number-theoretic axioms, the ontological axioms and (Pair).

Lemma 5.10. *Let $\mathscr{A}$ be any axiom of* BS *that is either a number-theoretic axiom, an ontological axiom or an instance of* (Pair)*. Then we can show that*

$$\mathsf{ACA}_0^+ \vdash |\mathscr{A}|_2.$$

Proof. We work in ACA_0^+. First, observe that for all representation trees R,

$$R \in^\star \mathrm{N}^\star \leftrightarrow \exists x(R = \{\langle\rangle, \langle 2x+1\rangle\}). \tag{5.1}$$

This is already provable in ACA_0. By (5.2), the claim follows directly by definition if $\mathscr{A}$ is a number-theoretic axiom of BS. Next, suppose that $\mathscr{A}$ is an ontological axiom. For (O1) we note that two representation trees S, T code the same object iff they either both code a set and $S =^\star T$, or they both code a natural number and $S = T$. Of course, we also have $S =^\star T$ in the latter case. The claim now follows from Theorem 4.19 and Example 4.6. Looking at Definition 5.1, the translations of (O2), (O3) are trivially valid. (O4), (O5), (O7) follow from (5.2). For (O6) we use in addition that $R \notin^\star \{\langle\rangle, \langle 2x+1\rangle\}$ for all x and representation trees R, cf. Example 4.6. It remains to deal with (Pair). To this end, let S, T be representation trees. By Theorem 4.18, let Y such that $\mathrm{Iso}(Y, S \oplus T)$. Using Theorem 4.19 we immediately obtain that for all representation trees R,

$$R \in^\star S \oplus T \leftrightarrow R =^\star S \vee R =^\star T.$$

Hence, $S \oplus T$ properly codes the set consisting of the sets coded by S and T. This finishes the proof. $\square$

Lemma 5.11. *For every Rudimentary Closure axiom $\mathscr{A}$ of BS^0,*

$$\mathsf{ACA}_0^+ \vdash |\mathscr{A}|_2.$$

Proof. In the following we work in ACA_0^+. We start with some preliminary remarks. Let S, T be representation trees that code sets. In particular, if $\langle x\rangle \in S$ or $\langle x\rangle \in T$, then x must be even. Suppose that S and T represent, respectively, the sets u and v. We have to show that there exist representation trees coding the sets $u \setminus v$, $\bigcup u$, $u \times v$, $\mathrm{dom}(u)$, $\in\restriction u$, u^{-1}, and

$$\begin{gathered}
\{\langle k, \langle l, m\rangle\rangle : \langle k, l\rangle \in u \wedge m \in v\}, \\
\{\langle k, \langle m, l\rangle\rangle : \langle k, l\rangle \in u \wedge m \in v\}, \\
\{y : \exists x(x \in u \wedge y = v'' \{x\})\}.
\end{gathered}$$

In order to show that the given trees properly represent the respective sets, we will tacitly apply Lemma 4.10, Theorem 4.18 and Theorem 4.19. Given

$\sigma \in \mathrm{Seq}$ and number n, we use notations such as $\langle (\sigma)_i : i > n \rangle$ to denote the sequence

$$\langle (\sigma)_{n+1}, (\sigma)_{n+2}, \ldots, (\sigma)_{\mathrm{lh}(\sigma)-1} \rangle,$$

and so on. We now go through all cases.

$u \setminus v$: Let Y such that $\mathrm{Iso}(Y, S \oplus T)$. To code $u \setminus v$ we first identify equal elements in u and v, i.e., we let

$$M := \{x : \exists y (\langle \langle 0, x \rangle, \langle 2, y \rangle \rangle \in Y)\}.$$

M will be used to identify child nodes of $\langle\rangle$ in S that correspond to objects in $u \cap v$. In concrete terms, we define the representation tree

$$S \setminus T := \{\sigma \in S : (\sigma)_0 \notin M\},$$

i.e., we remove branches in S that code objects already coded by T. We claim that $S \setminus T$ represents $u \setminus v$. To this end we have to show that

$$R \in^\star S \setminus T \leftrightarrow R \in^\star S \wedge R \notin^\star T, \tag{5.2}$$

where R is an arbitrary representation tree. By definition of M and Lemma 4.13 we can infer $(\forall x \notin M)(S^{\langle x \rangle} \notin^\star T \wedge S^{\langle x \rangle} = (S \setminus T)^{\langle x \rangle})$ and

$$R \in^\star S \setminus T \leftrightarrow (\exists x \notin M)(R =^\star S^{\langle x \rangle}).$$

From this the claim follows.

$\bigcup u$: To code $\bigcup u$ we proceed as follows. We transform S into the representation tree S_2 such that for all $\tau, \rho \in S$:

- $\mathrm{lh}(\tau) = 2 \to (\tau)_1$ is even,
- $\mathrm{lh}(\tau), \mathrm{lh}(\rho) \geq 2 \to (\tau)_1 \neq (\rho)_1$.

Using S_2 we can then obtain a representation tree $\bigcup S$ coding $\bigcup u$. The details are as follows. S_2 is constructed in two steps. First, we look at all $\tau \in S$ satisfying

$$\mathrm{lh}(\tau) = 2 \wedge (\tau)_1 \text{ is odd},$$

and replace these with, respectively, the sequences $\langle (\tau)_0 \rangle$, i.e., the last node of each such τ is deleted. We denote the resulting representation tree by S_1. In the second step we replace all $\sigma \in S_1$ by the sequence σ^* of the same length, where for all $i < \mathrm{lh}(\sigma)$, $(\sigma^*)_i = (\sigma)_i$ if $i \neq 1$, and

$$(\sigma^*)_1 = 2 \cdot \langle (\sigma)_0, (\sigma)_1 \rangle.$$

We let S_2 denote the resulting representation tree. S_2 exists by (ACA). Note that the order in which these step are carried out does not matter. To code $\bigcup u$ we can now delete the first element in all sequences of S_2, i.e., we define

$$\bigcup S := \{\tau \in \mathrm{Seq} : \exists x(\langle x \rangle * \tau \in S_2)\}.$$

It is straightforward to show that $\bigcup S$ is a representation tree. Intuitively, the definition of S_2 prepares S before the deletion of all child nodes of $\langle\rangle$ in S. Caution is required for two reasons. First, natural numbers contained in the set u coded by S must be replaced by $\emptyset$. This does not affect $\bigcup u$. Moreover, all grandchild nodes of $\langle\rangle$ in S must get a different label before the child nodes of $\langle\rangle$ are deleted. Note that $\bigcup S = \{\langle\rangle\}$ if S codes a natural number or a set containing only natural numbers. It remains to show that $\bigcup S$ properly represents $\bigcup u$, i.e., for all representation trees

$$R \in^\star \bigcup S \leftrightarrow \exists Q(\mathrm{Rep}(Q) \wedge R \in^\star Q \wedge Q \in^\star S). \tag{5.3}$$

By construction of $\bigcup S$ we get for w, x with $\langle 2w, 2x \rangle \in S$,

$$(\bigcup S)^{\langle 2\langle 2w, 2x \rangle\rangle} = S^{\langle 2w, 2x \rangle}.$$

By Lemma 4.13 we have $S^{\langle 2w,2x \rangle} \in^\star S^{\langle 2w \rangle}$ and $S^{\langle 2w,2x \rangle} \in^\star \bigcup S$. With the help of Theorem 4.19 it is now straightforward to verify (5.3).

$u \times v$: Given representation trees R_1, R_2, we define $\langle R_1, R_2 \rangle$ to be the representation tree consisting of the sequences

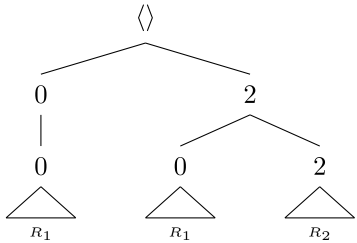
$$\langle\rangle, \langle 0 \rangle, \langle 0, 0 \rangle, \langle 2 \rangle, \langle 2, 0 \rangle, \langle 2, 2 \rangle,$$

plus all sequences of the form

$$\langle 0, 0 \rangle * \rho_1, \langle 2, 0 \rangle * \rho_1 \text{ and } \langle 2, 2 \rangle * \rho_2$$

for all $\rho_1 \in R_1, \rho_2 \in R_2$. By Theorem 4.19 one can easily see that $\langle R_1, R_2 \rangle$ codes the Kuratowski pair of the objects coded by R_1 and R_2. $\langle R_1, R_2 \rangle$

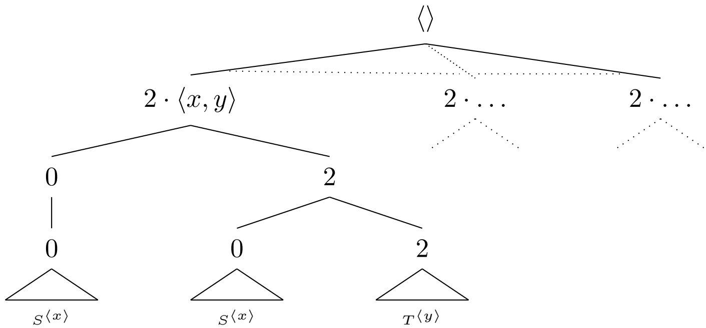
can be sketched as follows:



To represent $u \times v$ we define $S \times T$ to be the representation tree consisting of $\langle\rangle$ plus all $\sigma \in \mathrm{Seq}$ such that there exists $\langle x\rangle \in S$ and $\langle y\rangle \in T$ with

$$(\sigma)_0 = 2 \cdot \langle x, y\rangle \wedge \langle(\sigma)_i : i > 0\rangle \in \langle S^{\langle x\rangle}, T^{\langle y\rangle}\rangle.$$

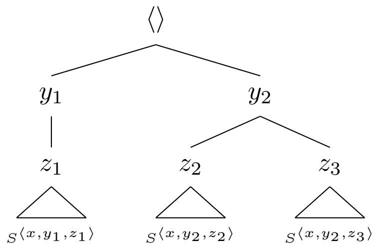
Recall that x, y must be even. Demanding $(\sigma)_0$ to be even makes sure that $S \times T$ codes the set consisting of the objects coded by $(S \times T)^{2\langle x,y\rangle}$ for all x, y with $\langle x\rangle \in S$ and $\langle y\rangle \in T$. Since $(S \times T)^{2\langle x,y\rangle} = \langle S^{\langle x\rangle}, T^{\langle y\rangle}\rangle$, $S \times T$ codes $u \times v$. If S or T codes $\emptyset$, $S \times T = \{\langle\rangle\}$. $S \times T$ can be depicted as follows:



Exploiting Lemma 4.13 and Theorem 4.19, and proceeding similarly as in the previous cases, it is easily verified that $S \times T$ properly represents $u \times v$.

dom(u): Some additional notions are required. Let X such that $\mathrm{Iso}(X, S)$ holds. We say that x codes a pair in S, written $\mathrm{pair}(x, S)$, if $\langle x\rangle \in S$, and there exist even numbers y_1, y_2, z_1, z_2, z_3 with $y_1 \neq y_2$,

$z_2 \neq z_3$, $\langle\langle x, y_1, z_1\rangle, \langle x, y_2, z_2\rangle\rangle \in X$ such that $S^{\langle x\rangle}$ is of the form



(5.4)

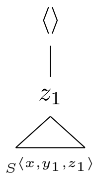
Recall that x is even by assumption. The idea is that $\mathrm{pair}(x, S)$ states that the object coded by $S^{\langle x\rangle}$ is a pair. X identifies subtrees representing the same object since by Lemma 4.13 we have

$$S^{\langle x,y_1,z_1\rangle} =^\star S^{\langle x,y_2,z_2\rangle}.$$

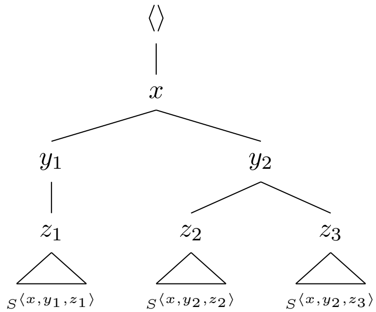
By Theorem 4.19 it is clear that $\mathrm{pair}(x, S)$ has the intended meaning. Note that $\mathrm{pair}(x, S)$ is an arithmetical assertion. Now, in order to define the representation tree $\mathrm{dom}(S)$ coding $\mathrm{dom}(u)$, we proceed in two steps. First, we let M be the set consisting of all pairs $\langle x, \sigma\rangle$ with $\sigma \in \mathrm{Seq}$ such that

$$\langle x, y_1\rangle * \sigma \in S \wedge \mathrm{pair}(x, S) \wedge \langle\langle x, y_1, z_1\rangle, \langle x, y_2, z_2\rangle\rangle \in X$$

for some y_1, y_2, z_1, z_2, z_3. Intuitively, M is the disjoint union of all trees



that are cut out from the following subtree of S,

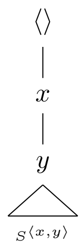


where $\mathrm{pair}(x, S)$ and $\langle\langle x, y_1, z_1\rangle, \langle x, y_2, z_2\rangle\rangle \in X$. Given x, the numbers y_1, y_2, z_1, z_2, z_3 are uniquely determined by definition. Secondly, we define $\mathrm{dom}(S)$ to consist of $\langle\rangle$ plus all $\sigma^* \in \mathrm{Seq}$ satisfying

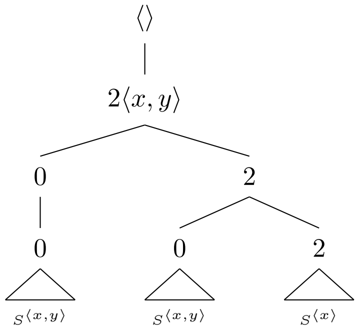
$$\begin{aligned}&\exists x, \sigma(\langle x, \sigma\rangle \in M \wedge \mathrm{lh}(\sigma^*) = \mathrm{lh}(\sigma) \wedge \\ &\qquad (\forall i < \mathrm{lh}(\sigma))((\sigma)_i = (\sigma^*)_i \leftrightarrow i \neq 0 \wedge (\sigma^*)_0 = x)).\end{aligned}$$

The idea of the above construction is as follows. M is the disjoint union of trees coding the wanted elements. However, to put all these trees together into a proper representation tree we have to relabel nodes in order to avoid conflicts. The above definition ensures that. Using Lemma 4.13 and Theorem 4.19 we can therefore verify that $\mathrm{dom}(S)$ properly codes $\mathrm{dom}(u)$.

$\in\restriction u$: The intuition behind coding $\in\restriction u$ is to consider every subtree of S of the form,



with x, y being even, and replace that tree by

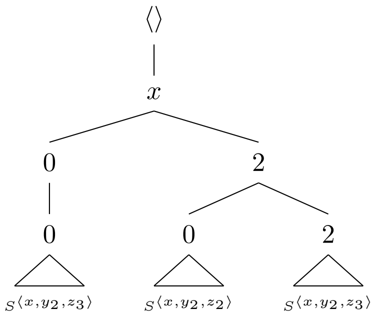


Finally, to form $\in\restriction S$, all these newly formed trees are glued together. More formally, $\in\restriction S$ consists of $\langle\rangle$ plus all sequences σ such that

$$(\sigma)_0 = 2\langle x, y\rangle \wedge \langle(\sigma)_i : i > 0\rangle \in \langle S^{\langle x,y\rangle}, S^{\langle x\rangle}\rangle,$$

where x, y are such that y is even and $\langle x, y\rangle \in S$. The construction makes sure that subtrees of the form $S^{\langle x\rangle}$ that code $\emptyset$ or natural numbers are discarded. By Lemma 4.13 and Theorem 4.19 we see that $(\in\restriction S)^{\langle 2\langle x,y\rangle\rangle}$ represents the correct set for each x, y as above. Hence, $\in\restriction S$ properly codes $\in\restriction u$.

u^{-1}: Put X such that $\mathrm{Iso}(X, S)$ holds. To construct S^{-1} coding u^{-1}, we let x such that $\mathrm{pair}(x, S)$ holds and consider the following tree,



where, by pair(x, S), y_1, y_2, z_1, z_2, z_3 are uniquely determined such that

$$\langle\langle x, y_1, z_2\rangle, \langle x, y_2, z_2\rangle\rangle \in X.$$

Then S^{-1} is obtained by glueing together the above trees for all x with pair(x, S). The idea is as follows. If $S^{\langle x\rangle}$ codes a pair, then the above tree codes exactly the inverted pair. All other trees of the form $S^{\langle x\rangle}$ are discarded. More formally, S^{-1} consists of $\langle\rangle$ plus all $\sigma \in \mathrm{Seq}$ such that there exist $x, y_1, y_2, z_1, z_2, z_3$ with

$$\mathrm{pair}(x, S) \wedge \langle\langle x, y_1, z_2\rangle, \langle x, y_2, z_2\rangle\rangle \in X$$

and

$$(\sigma)_0 = x \wedge \langle(\sigma)_i : i > 0\rangle \in \langle S^{\langle x,y_2,z_3\rangle}, S^{\langle x,y_2,z_2\rangle}\rangle.$$

The above intuition can be made precise using Lemma 4.13 and Theorem 4.19, therefore S^{-1} properly codes u^{-1}.

$\{\langle k, \langle l, m\rangle\rangle : \langle k, l\rangle \in u \wedge m \in v\}$: Let M be the set of all numbers of the form $\langle x, \langle y_2, z_2, z_3\rangle\rangle$ such that

$$\mathrm{pair}(x, S) \wedge z_2 \neq z_3 \wedge \langle y_2, z_2\rangle, \langle y_2, z_3\rangle \in S.$$

Let W be the set of all sequences σ such that there exists w, x, y_2, z_2, z_3 with $\langle w\rangle \in T$ and $\langle x, \langle y_2, z_2, z_3\rangle\rangle \in M$ so that

$$(\sigma)_0 = 2\langle w, x\rangle \wedge \langle(\sigma)_i : i > 0\rangle \in \langle S^{\langle x,y_2,z_2\rangle}, \langle S^{\langle x,y_2,z_3\rangle}, T^{\langle w\rangle}\rangle\rangle.$$

For better readability, the indices are chosen analogously to (5.4). As in the previous cases, we can easily convince ourselves that W is a representation tree coding the desired set.

$\{\langle k, \langle m, l\rangle\rangle : \langle k, l\rangle \in u \wedge m \in v\}$: We proceed similar as in the previous case and let W be the set of all $\sigma \in \mathrm{Seq}$ such that there exists w, x, y_2, z_2, z_3 with

$$\langle w\rangle \in T \wedge \mathrm{pair}(x, S) \wedge z_2 \neq z_3 \wedge \langle y_2, z_2\rangle, \langle y_2, z_3\rangle \in S,$$

where

$$(\sigma)_0 = 2\langle w, x\rangle \wedge \langle(\sigma)_i : i > 0\rangle \in \langle S^{\langle x,y_2,z_3\rangle}, \langle S^{\langle x,y_2,z_2\rangle}, T^{\langle w\rangle}\rangle\rangle.$$

As before, one can easily verify that W properly codes the desired set.

$\{y : \exists x(x \in u \wedge y = v'' \{x\})\}$: Define the sets Y, Z such that, respectively, $\mathrm{Iso}(Y, T)$ and $\mathrm{Iso}(Z, S \oplus T)$. Now, let M be the set of all pairs $\langle x, \langle w, y_2, z_3\rangle\rangle$ satisfying

$$\langle x\rangle \in S \wedge \langle w, y_2, z_3\rangle \in T \wedge \mathrm{pair}(w, T)$$

such that there exists y_1, z_1, z_2 with $y_2 \neq y_1$, $z_3 \neq z_2$ and

$$\langle\langle w, y_1, z_1\rangle, \langle w, y, z_2\rangle\rangle \in Y \wedge \langle\langle 0, x\rangle, \langle 2, w, y_2, z_3\rangle\rangle \in Z.$$

Intuitively, $\langle x, \langle w, y_2, z_3\rangle\rangle \in M$ iff $T^{\langle w\rangle}$ codes a pair with first component isomorphic to $S^{\langle x\rangle}$. The numbers y_2, z_3 serve to identify the second component of the coded pair. Recall that y_1, y_2, z_1, z_2, z_3 are uniquely determined by w. M provides all data necessary to obtain the desired set, denoted W. Namely, we let W consist of all $\sigma \in \mathrm{Seq}$ such that

$$\exists x, w, y_2, z_3(\langle x, \langle w, y_2, z_3\rangle\rangle \in M \wedge (\sigma)_0 = x \wedge \langle(\sigma)_i : i > 0\rangle \in T^{\langle w, y_2, z_3\rangle}).$$

It is straightforward to verify that W is a representation tree. For each subtree of the form $S^{\langle x\rangle}$, there exists a subtree $W^{\langle x\rangle}$ coding $v'' \{x\}$. Moreover, W does not contain any other subtrees. Thus, W codes the desired set. Similarly as before, this is straightforward to check. As in the previous two cases, the indices are chosen in accordance with (5.4). $\square$

5.3. Interpreting $(\Delta_0\text{-I}_\mathrm{N})$, $(\mathcal{L}_\mathrm{s}\text{-I}_\mathrm{N})$ and $(\mathcal{L}_\mathrm{s}\text{-I}_\in)$

Next, we turn to the induction schemas $(\Delta_0\text{-I}_\mathrm{N})$, $(\mathcal{L}_\mathrm{s}\text{-I}_\mathrm{N})$ and $(\mathcal{L}_\mathrm{s}\text{-I}_\in)$. We introduce an additional notion to facilitate the discussion. Let (Schem) be an arbitrary axiom schema of $\mathcal{L}_\mathrm{s}$. Then $|(\mathrm{Schem})|_2$ is the collection of all translations of (Schem). More precisely, it is the collection of all $\mathcal{L}_2$ formulas of the form $|\mathscr{F}|_2$, where $\mathscr{F}$ is some instance of (Schem). As mentioned before, it turns out that $|(\Delta_0\text{-I}_\mathrm{N})|_2$, $|(\mathcal{L}_\mathrm{s}\text{-I}_\mathrm{N})|_2$ and $|(\mathcal{L}_\mathrm{s}\text{-I}_\in)|_2$ are all provable in ACA. Moreover, $|(\Delta_0\text{-I}_\mathrm{N})|_2$ is already derivable in $\Sigma^1_1\text{-}\mathsf{AC}^+_0$. In the following we will cover all details and establish these assertions. We then have all the ingredients to embed BS^0 into $\Sigma^1_1\text{-}\mathsf{AC}^+_0$, and $\mathsf{BS}^1, \mathsf{BS}$ into ACA.

Lemma 5.12. *We have the following:*

(a) $\Sigma^1_1\text{-}\mathsf{AC}^+_0 \vdash |(\Delta_0\text{-I}_\mathrm{N})|_2$.

(b) $\mathsf{ACA} \vdash |(\mathcal{L}_\mathrm{s}\text{-I}_\mathrm{N})|_2$.

(c) $\mathsf{ACA} \vdash |(\mathcal{L}_\mathrm{s}\text{-I}_\in)|_2$.

Proof. (a): Suppose that $\mathscr{A}[w, \vec{u}]$ is a Δ_0 formula with free object variables among $w, u_1, \ldots, u_n$. We look at the following instance of $(\Delta_0\text{-I}_\mathrm{N})$:

$$\mathscr{A}[0, \vec{u}] \wedge (\forall m, n \in \mathrm{N})(\mathscr{A}[m, \vec{u}] \wedge \mathrm{R}_{\mathrm{Succ}}(m, n) \to \mathscr{A}[n, \vec{u}]) \\ \to (\forall n \in \mathrm{N})\mathscr{A}[n, \vec{u}].$$

Working in $\Sigma^1_1\text{-}\mathsf{AC}^+_0$, suppose $\mathrm{Rep}(u^\star_1, \ldots, u^\star_n)$, $\mathscr{A}^\star[0^\star, \vec{u}]$ and that for all representation trees $S, T \in \mathrm{N}^\star$:

$$\mathscr{A}^\star[S, \vec{u}^\star] \wedge \mathrm{R}_{\mathrm{Succ}}(S, T)^\star \to \mathscr{A}^\star[T, \vec{u}^\star]. \tag{5.5}$$

We have to show that for all representation trees $R \in^\star \mathrm{N}$, it holds that $\mathscr{A}^\star(R, \vec{u}^\star)$. The idea is to apply Δ^1_1 induction to a suitable formula. Since $(\Delta^1_1\text{-CA})$ is provable in $\Sigma^1_1\text{-}\mathsf{AC}_0$, a subsystem of $\Sigma^1_1\text{-}\mathsf{AC}^+_0$, induction for Δ^1_1 formulas is available. By Corollary 5.6 we can asssume that $\mathscr{A}^\star(R, \vec{u}^\star)$ is Δ^1_1. By Theorem 4.19(e) and Example 4.6 we obtain that for all representation trees R,

$$R \in^\star \mathrm{N} \leftrightarrow \exists m(R = \{\langle\rangle, \langle 2m+1\rangle\}).$$

Given a representation $R \in^\star \mathrm{N}$, we use $\#(R)$ to denote the unique m such that $\langle 2m+1\rangle \in R$. It follows that for all representation trees $S, T \in^\star \mathrm{N}$,

$$\mathrm{R}_{\mathrm{Succ}}(S, T)^\star \leftrightarrow \#(S) + 1 = \#(T). \tag{5.6}$$

Next, define the formula

$$\mathscr{B}[n, \vec{u}^\star] :\equiv \mathscr{A}^\star[\{\langle\rangle, \langle 2n+1\rangle\}, \vec{u}^\star].$$

Note that $\mathscr{B}[n, \vec{u}^\star]$ is Δ^1_1. By what we showed above we have for all representation trees $R \in^\star \mathrm{N}^\star$,

$$\mathscr{A}^\star[R, \vec{u}^\star] \leftrightarrow \mathscr{B}^\star[\#(R), \vec{u}^\star]. \tag{5.7}$$

$\mathscr{A}^\star[0^\star, \vec{u}]$ is equal to $\mathscr{B}^\star[0, \vec{u}]$. Looking at (5.5) and (5.6) we obtain

$$\mathscr{B}^\star[m, \vec{u}^\star] \wedge n = m + 1 \to \mathscr{B}^\star[n, \vec{u}^\star].$$

Hence, applying Δ^1_1 induction to $\mathscr{B}^\star[n, \vec{u}^\star]$ we obtain $\forall n \mathscr{B}[n, \vec{u}^\star]$. Using (5.7) we can conclude the proof.
(b): The assertion follows in basically the same manner as (a). In fact, we don't need to rely on Corollary 5.6. Instead of Δ^1_1 induction, we apply induction for $\mathcal{L}_2$ formulas, which is available in ACA.
(c): Let $\mathscr{A}[w, \vec{u}]$ be any $\mathcal{L}_\mathrm{s}$ formula with free object variables $w, u_1, \dots, u_n$. We consider the following instance of ($\mathcal{L}_\mathrm{s}$-$\mathrm{I}_\in$):

$$\forall x((\forall y \in x)\mathscr{A}[y, \vec{u}] \to \mathscr{A}[x, \vec{u}]) \to \forall x \mathscr{A}[x, \vec{u}].$$

Working in ACA, suppose that $\mathrm{Rep}(u_1^\star, \dots, u_n^\star)$ and for all representation trees T,

$$\forall S(\mathrm{Rep}(S) \wedge S \in^\star T \to \mathscr{A}^\star[S, \vec{u}^*]) \to \mathscr{A}^\star[T, \vec{u}^*]. \tag{5.8}$$

We want to show that $\mathscr{A}^\star[T, \vec{u}^*]$ for all representation trees T. Since all representation trees have finite length, we can achieve this by a suitable induction. Put

$$\mathscr{B}[k, \vec{u}^*] :\equiv \forall T(\mathrm{Rep}(T) \wedge k = \max\{\mathrm{lh}(\sigma) : \sigma \in T\} \to \mathscr{A}^\star[T, \vec{u}^*]).$$

It suffices to show $\forall k \mathscr{B}[k, \vec{u}^*]$. We proceed by $<$-induction. The base cases $k = 0, 1$ are immediate as (5.8) is vacuously true. For the induction step we fix $k > 1$ and assume $\mathscr{B}[l, \vec{u}^*]$ for all $l < k$. Let T be a representation tree such that $k + 1 = \max\{\mathrm{lh}(\sigma) : \sigma \in T\}$. In particular, T does not code the empty set. Let S be a representation tree such that $S \in^* T$. By Lemma 4.21 we have

$$\max\{\mathrm{lh}(\sigma) : \sigma \in S\} < \max\{\mathrm{lh}(\sigma) : \sigma \in T\}.$$

By the induction hypothesis we therefore get

$$\forall S(\mathrm{Rep}(S) \wedge S \in^\star T \to \mathscr{A}^\star[S, \vec{u}^*]).$$

Using (5.8) we obtain

$$\mathscr{A}^\star[T, \vec{u}^*].$$

Altogether, we derived $|\mathscr{A}|_2[T, \vec{u}^*]$, which concludes the proof. $\square$

Theorem 5.13. *Let* $\mathscr{A}$ *be any* $\mathcal{L}_\mathrm{s}$ *formula. Then we have*

(a) $\mathsf{BS}^0 \vdash \mathscr{A}$ *implies* $\Sigma^1_1\text{-}\mathsf{AC}^+_0 \vdash |\mathscr{A}|_2$.

(b) $\mathsf{BS}^1 \vdash \mathscr{A}$ *implies* $\mathsf{ACA} \vdash |\mathscr{A}|_2$.

(c) $\mathsf{BS} \vdash \mathscr{A}$ *implies* $\mathsf{ACA} \vdash |\mathscr{A}|_2$.

Proof. Note that (b) follows immediately from (c). For each assertion it suffices to prove the claim for all axioms of the respective set theory. This is taken care of by Lemma 5.10, Lemma 5.11 and Lemma 5.12. $\square$

6. Adding Σ and Π reduction

In Part I we studied several principles over ACA_0 and proved their equivalence to (ATR). Among these principles, Σ^1_1 and Π^1_1 reduction have very natural set-theoretic counterparts. It is the goal of this section to study these within BS and its subsystems.

6.1. Π and Σ reduction

In this section we will extend BS and its subsystem with set-theoretic analogues of Π^1_1 and Σ^1_1 reduction, cf. section 2.2.

The schemas of Π and Σ reduction are given as follows.

Π and Σ reduction.

For all Σ formulas $\mathscr{A}(x)$ and all Π formulas $\mathscr{B}(x)$, the schema (Π-Red) comprises all formulas of the form

$$\begin{aligned}&(\forall x \in a)(\mathscr{A}(x) \to \mathscr{B}(x)) \to \\ &\quad \exists y((\forall x \in a)(\mathscr{A}(x) \to x \in y) \land (\forall x \in y)(x \in a \land \mathscr{B}(x))).\end{aligned} \tag{Π-Red}$$

Similarly, the schema (Σ-Red) consists of all formulas

$$\begin{aligned}&(\forall x \in a)(\mathscr{B}(x) \to \mathscr{A}(x)) \to \\ &\quad \exists y((\forall x \in a)(\mathscr{B}(x) \to x \in y) \land (\forall x \in y)(x \in a \land \mathscr{A}(x))).\end{aligned} \tag{Σ-Red}$$

In fact, the set-theoretic reduction principles introduced above are rather the counterparts of Π^1 and Σ^1 reduction in $\mathcal{L}_2$, henceforth termed (Π^1-Red) and (Σ^1-Red), respectively, cf. also Definition 2.1. These schemas are defined exactly as (Π^1_1-Red) and (Σ^1_1-Red), but with Π^1 and Σ^1 formulas playing the role of, respectively, Π^1_1 and Σ^1_1 formulas. By Theorem 2.2 and Corollary 2.4 we obtain that:

- Over Σ^1_1-AC_0, (Σ^1_1-Red) and (Σ^1-Red) are equivalent,
- Over ACA_0, (Π^1_1-Red) and (Π^1-Red) are equivalent.

Thus, on the side of second order arithmetic, we have equivalences in all interesting cases, as will become clear below. However, on the set-theoretic side we have to stick to the schemas (Π-Red) and (Σ-Red) since they offer more flexibility in our context. In particular, they both yield Δ separation over BS^0, i.e., for all Σ formulas $\mathscr{A}(x)$ and Π formulas $\mathscr{B}(x)$,

$$(\forall x \in a)(\mathscr{A}(x) \leftrightarrow \mathscr{B}(x)) \rightarrow \exists y \forall x(x \in y \leftrightarrow (x \in a \wedge \mathscr{A}(x))).$$

From that, lower bounds for (Σ-Red) over BS^0 and BS^1 can be read off immediately.

Lemma 6.1. *We have the following inclusions:*

$$\Delta^1_1\text{-}\mathsf{CA}_0 \subseteq \mathsf{BS}^0 + (\Sigma\text{-Red}) \quad \textit{and} \quad \Delta^1_1\text{-}\mathsf{CA} \subseteq \mathsf{BS}^1 + (\Sigma\text{-Red}).$$

Proof. Recall that we can view $\mathcal{L}_2$ as a sublanguage of $\mathcal{L}_\mathrm{s}$ by mapping any $\mathcal{L}_2$ formula $\mathscr{A}$ to $\mathscr{A}^\circ$, cf. section 4.2. Note that (Δ^1_1-CA) instances become instances of Δ separation under this translsation. Thus, the assertion follows directly by the above discussion and Theorem 4.4. $\square$

In case (Π-Red) is available, we can do more.

Lemma 6.2. *We have the following inclusions:*

$$\mathsf{ATR}_0 \subseteq \mathsf{BS}^0 + (\Pi\text{-Red}) \quad \textit{and} \quad \mathsf{ATR} \subseteq \mathsf{BS}^1 + (\Pi\text{-Red}).$$

Proof. It is clear that, modulo our translation of $\mathcal{L}_2$ into $\mathcal{L}_\mathrm{s}$, instances of (Π^1_1-Red) become special cases of (Π-Red). Since (ATR) is equivalent to (Π^1_1-Red) over ACA_0, cf. Theorem 2.2, the assertion follows immediately by Theorem 4.4. $\square$

We will show that the bounds given above are sharp w.r.t. proof-theoretic strength. In order to deal with (Σ-Red) we establish that, working in Σ^1_1-AC_0, (Σ^1-Red) is valid on the collection of representation trees.

Lemma 6.3. *Let $\mathscr{A}(X)$ be a Σ^1 formula of $\mathcal{L}_2$ and $\mathscr{B}(X)$ a Π^1 formula of $\mathcal{L}_2$. Working in Σ^1_1-AC_0, assume that S is a representation tree such that*

$$\forall x(\langle 2x \rangle \in S \wedge \mathscr{B}(S^{\langle 2x \rangle}) \rightarrow \mathscr{A}(S^{\langle 2x \rangle})).$$

Then there exists a representation tree T such that:

(a) $\forall x(\langle 2x\rangle \in S \wedge \mathscr{B}(S^{\langle 2x\rangle}) \to \langle 2x\rangle \in T \wedge S^{\langle 2x\rangle} = T^{\langle 2x\rangle})$,

(b) $\forall x(\langle 2x\rangle \in T \to \langle 2x\rangle \in S \wedge S^{\langle 2x\rangle} = T^{\langle 2x\rangle} \wedge \mathscr{A}(T^{\langle 2x\rangle}))$.

Proof. Suppose we work in Σ^1_1-AC_0. By Theorem 2.2 and Corollary 2.4, (Σ^1-Red) is at our disposal. We put

$$\mathscr{A}'(z) :\equiv \exists x(z = 2x \wedge \mathscr{A}(S^{\langle 2x\rangle})),$$
$$\mathscr{B}'(z) :\equiv \exists x(z = 2x \wedge \langle 2x\rangle \in S \wedge \mathscr{B}(S^{\langle 2x\rangle})).$$

By assumption, $\mathscr{A}'(z)$ is Σ^1 and $\mathscr{B}'(z)$ is Π^1. Moreover, we have

$$\forall z(\mathscr{B}'(z) \to \mathscr{A}'(z)).$$

Applying (Σ^1-Red) with S as parameter gives a set Y such that

$$\forall z(z \in Y \to \mathscr{A}'(z)) \wedge \forall z(\mathscr{B}'(z) \to z \in Y). \tag{6.1}$$

By (ACA) we put

$$T := \{\langle\rangle\} \cup \{\sigma \in S : 1 \leq \mathrm{lh}(\sigma) \wedge (\sigma)_0 \in Y\}.$$

We claim that T is a representation tree with the desired properties. Without loss of generality we can assume that S codes a set. Then it is clear that T is a representation tree because S is one. We have to show that (a) and (b) hold. For (a), let x such that $\langle 2x\rangle \in S$ and $\mathscr{B}(S^{\langle 2x\rangle})$. Then we have $\mathscr{B}'(2x)$, hence $2x \in Y$ by (6.1). By definition of T, it follows that $\langle 2x\rangle \in T$ and $S^{\langle 2x\rangle} = T^{\langle 2x\rangle}$. This shows (a). For (b), consider x satisfying $\langle 2x\rangle \in T$. By definition of T, this immediately yields $\langle 2x\rangle \in S$ and $S^{\langle 2x\rangle} = T^{\langle 2x\rangle}$. Moreover, we have $\langle 2x\rangle \in Y$, hence by (6.1) we also get $\mathscr{A}'(2x)$, i.e., $\mathscr{A}(S^{\langle 2x\rangle})$. This amounts to (b) and concludes the proof. □

In order to take care of (Π-Red) we establish (Π^1-Red) on the class of representation trees within ATR_0. The proof follows the same pattern as above. Together with Lemma 5.5, we then have the necessary tools to interpret (Σ-Red) and (Π-Red) properly within, respectively, Σ^1_1-AC_0 and ATR_0.

Lemma 6.4. *Let* $\mathscr{A}(X)$ *be a* Σ^1 *formula of* $\mathcal{L}_2$ *and* $\mathscr{B}(X)$ *a* Π^1 *formula of* $\mathcal{L}_2$*. Working in* ATR_0*, suppose that* S *is a representation tree such that*

$$\forall x(\langle 2x\rangle \in S \wedge \mathscr{A}(S^{\langle 2x\rangle}) \to \mathscr{B}(S^{\langle 2x\rangle})).$$

Then there exists a representation tree T *such that:*

(a) $\forall x(\langle 2x\rangle \in S \wedge \mathscr{A}(S^{\langle 2x\rangle}) \to \langle 2x\rangle \in T \wedge S^{\langle 2x\rangle} = T^{\langle 2x\rangle})$,

(b) $\forall x(\langle 2x\rangle \in T \to \langle 2x\rangle \in S \wedge S^{\langle 2x\rangle} = T^{\langle 2x\rangle} \wedge \mathscr{B}(T^{\langle 2x\rangle}))$.

Proof. We work in ATR_0. Recall that we can make use of (Π^1-Red), cf. Theorem 2.2 and Corollary 2.4. We let

$$\begin{aligned}\mathscr{A}'(z) &:\equiv \exists x(z = 2x \wedge \langle 2x\rangle \in S \wedge \mathscr{A}(S^{\langle 2x\rangle})),\\ \mathscr{B}'(z) &:\equiv \exists x(z = 2x \wedge \mathscr{B}(S^{\langle 2x\rangle})).\end{aligned}$$

By assumption, $\mathscr{A}'(z)$ is Σ^1 and $\mathscr{B}'(z)$ is Π^1. Moreover, we have

$$\forall z(\mathscr{A}'(z) \to \mathscr{B}'(z)).$$

Applying (Π^1-Red) with S as parameter gives a set Y such that

$$\forall z(\mathscr{A}'(z) \to z \in Y) \wedge \forall z(z \in Y \to \mathscr{B}'(z)). \tag{6.2}$$

Using (ACA) we define

$$T := \{\langle\rangle\} \cup \{\sigma \in S : 1 \le \mathrm{lh}(\sigma) \wedge (\sigma)_0 \in Y\}.$$

We show that T has the desired properties. We can suppose that S codes a set. Clearly, T is then a proper representation tree. It remains to show that (a) and (b) hold. For (a) let x such that $\langle 2x\rangle \in S$ and $\mathscr{A}(S^{\langle 2x\rangle})$. Then we have $\mathscr{A}'(2x)$, hence $2x \in Y$ by (6.2). By definition of T, it follows that $\langle 2x\rangle \in T$ and $S^{\langle 2x\rangle} = T^{\langle 2x\rangle}$. This shows (a). To establish (b), let x such that $\langle 2x\rangle \in T$. By definition of T this yields $\langle 2x\rangle \in S$ and $S^{\langle 2x\rangle} = T^{\langle 2x\rangle}$. Furthermore, we have $\langle 2x\rangle \in Y$, therefore by (6.2) we also obtain $\mathscr{B}'(2x)$, i.e., $\mathscr{B}(S^{\langle 2x\rangle})$. This proves (b), finishing the proof. $\square$

6.2. Results

Except for the case $\mathsf{BS}^0 + (\Sigma\text{-Red})$, which is discussed in [BJ20], we can now determine the effect of adding $(\Sigma\text{-Red})$ and $(\Pi\text{-Red})$ to our basic set theories w.r.t. proof-theoretic strength.

Theorem 6.5. *For all formulas $\mathscr{A}$ of $\mathcal{L}_\mathrm{s}$ we have:*

(a) $\mathsf{BS} + (\Sigma\text{-Red}) \vdash \mathscr{A} \quad \Longrightarrow \quad \Sigma^1_1\text{-}\mathsf{AC} \vdash |\mathscr{A}|_2$.

(b) $\mathsf{BS}^0 + (\Pi\text{-Red}) \vdash \mathscr{A} \quad \Longrightarrow \quad \mathsf{ATR}_0 \vdash |\mathscr{A}|_2$.

(c) $\mathsf{BS} + (\Pi\text{-Red}) \vdash \mathscr{A} \quad \Longrightarrow \quad \mathsf{ATR} \vdash |\mathscr{A}|_2$.

Proof. For each assertion, it suffices to prove the claim for all axioms of the respective set theory. As in section 5.3, we use $|(\text{Schem})|_2$ to denote the translation of an axiom schema (Schem) of $\mathcal{L}_\mathrm{s}$ into $\mathcal{L}_2$. By Lemma 5.5 we can ensure that our translation carries Σ and Π formulas of $\mathcal{L}_\mathrm{s}$ over to, respectively, Σ^1 and Π^1 formulas of $\mathcal{L}_2$ in the given contexts. We now prove all assertions. For (a) we use Theorem 5.13(c), which only leaves $(\Sigma\text{-Red})$. By Lemma 5.5(b) and Lemma 6.3, $\Sigma^1_1\text{-}\mathsf{AC}_0$ proves all instances of $|(\Sigma\text{-Red})|_2$. For (b) we note that $\Sigma^1_1\text{-}\mathsf{AC}^+_0$ is contained in ATR_0, cf. Theorem 2.2. By Theorem 5.13(a) we are only left with $|(\Pi\text{-Red})|_2$. This is provable in ATR_0 by Lemma 5.5(c) and Lemma 6.4. For (c) we employ Theorem 5.13(c) and deal with $|(\Pi\text{-Red})|_2$ as for (b). □

Corollary 6.6. *We have the following proof-theoretic equivalences:*

(a) $|\Sigma^1_1\text{-}\mathsf{AC}| = |\mathsf{BS}^1 + (\Sigma\text{-Red})| = |\mathsf{BS} + (\Sigma\text{-Red})|$.

(b) $|\mathsf{ATR}_0| = |\mathsf{BS}^0 + (\Pi\text{-Red})|$.

(c) $|\mathsf{ATR}| = |\mathsf{BS}^1 + (\Pi\text{-Red})| = |\mathsf{BS} + (\Pi\text{-Red})|$.

Proof. Recall that $|\Sigma^1_1\text{-}\mathsf{AC}| = |\Delta^1_1\text{-}\mathsf{CA}| = \varphi_{\varepsilon_0}(0)$, cf. [Buc05]. Referring to Lemma 5.9, the corollary is a direct consequence of Lemma 6.1, Lemma 6.2 and Theorem 6.5. □

For completness we mention the result regarding the proof-theoretic strength of $\mathsf{BS}^0 + (\Sigma\text{-Red})$ presented in [BJ20]. We omit the details as the proof deviates from our thematic focus. It involes a combination of partial cut elimination and the method of asymmetric interpretations. The main idea is to reduce suitable fragments of $\mathsf{BS}^0 + (\Sigma\text{-Red})$ to substructures that are provided in $\mathsf{\Sigma}^1_1\text{-}\mathsf{AC}_0$ by the class of all k-trees for some fixed (standard) natural number k.

Theorem 6.7. *The following proof-theoretic equivalences hold:*

$$|\mathsf{PA}| = |\mathsf{\Delta}^1_1\text{-}\mathsf{CA}_0| = |\mathsf{\Sigma}^1_1\text{-}\mathsf{AC}_0| = |\mathsf{BS}^0 + (\Sigma\text{-Red})| = \varepsilon_0.$$

Proof. See [BJ20], section 9.3.5. $\square$

7. Adding Axiom Beta

In this chapter we expand our setting to obtain theories linked to Simpson's $\mathsf{ATR}_0^{\mathsf{set}}$ and related theories. $\mathsf{ATR}_0^{\mathsf{set}}$ is formulated in a set-theoretic language without urelements. It builds upon a basic set theory, which includes the Axiom of Infinity, and in which the existence of the first limit ordinal ω is provable. $\mathsf{ATR}_0^{\mathsf{set}}$ is then obtained by adding the Axiom Beta and the Axiom of Countability. Adapted to our context, we introduce a slightly modified variant of BS^0, denoted BS^2, as our new base theory. Building on BS^2, the Axiom Beta is added to obtain the theory BETA_0. Here, we follow the nomenclature of subsystems of second order arithmetic. The Axiom of Countability is not included in BETA_0. It will be added contextually, whenever needed. Adding Countability leads to a theory equivalent to $\mathsf{ATR}_0^{\mathsf{set}}$. As in the previous chapter, we are then interested in the effect of adding reduction principles to BETA_0 and related theories.

7.1. The systems BETA_0 and BETA

Definition 7.1. BS^2 is the formal system in $\mathcal{L}_\mathrm{s}$ which consists of BS^0 plus $(\Delta_0\text{-I}_\in)$. Alternatively, BS^2 is obtained from BS^1 by adding $(\Delta_0\text{-I}_\in)$ and restricting $(\mathcal{L}_\mathrm{s}\text{-I}_\mathrm{N})$ to $(\Delta_0\text{-I}_\mathrm{N})$.

In order to introduce the Axiom Beta, the Axiom of Countability and the Axiom of Infinity, a couple of additional mathematical notions are needed, cf. [Sim09].

- $\mathrm{Rel}(r) \leftrightarrow r \subseteq \mathrm{rng}(r) \times \mathrm{dom}(r)$, i.e., r is a relation,
- $\mathrm{Fcn}(f) \leftrightarrow \mathrm{Rel}(f) \wedge \forall x, y_1, y_2(\langle x, y_1\rangle \in f \wedge \langle x, y_2\rangle \in f \to y_1 = y_2)$, i.e., f is a function,
- $\mathrm{Inj}(f) \leftrightarrow \mathrm{Fcn}(f) \wedge \forall x_1, x_2, y(\langle x_1, y\rangle \in f \wedge \langle x_2, y\rangle \in f \to x_1 = x_2)$, i.e., f is an injection,

- $s \approx t \leftrightarrow \exists f(\mathrm{Inj}(f) \wedge \mathrm{dom}(f) = s \wedge \mathrm{rng}(f) = t)$, i.e., s and t are equinumerous,
- $\mathrm{Trans}(t) \leftrightarrow \forall x, y((x \in y \wedge y \in t) \to x \in t)$, i.e., t is transitive,
- $\mathrm{Ord}(t) \leftrightarrow \mathrm{Trans}(t) \wedge \forall x, y((x \in t \wedge y \in t) \to (x \in y \vee x = y \vee y \in x))$, i.e., t is an ordinal,
- $\mathrm{Succ}(t) \leftrightarrow \mathrm{Ord}(t) \wedge \exists v(t = v \cup \{v\})$, i.e., t is a successor ordinal,
- $\mathrm{Lim}(t) \leftrightarrow \mathrm{Ord}(t) \wedge t \neq \emptyset \wedge \neg \mathrm{Succ}(t)$, i.e., t is a limit ordinal,
- $\mathrm{FinOrd}(t) \leftrightarrow \mathrm{Ord}(t) \wedge \forall v(v \in t \cup \{t\} \to (v = \emptyset \vee \mathrm{Succ}(v)))$, i.e., t is a finite ordinal,
- $\mathrm{Fin}(t) \leftrightarrow \exists v(t \approx v \wedge \mathrm{FinOrd}(v))$, i.e., t is a finite set,
- $\mathrm{HFin}(t) \leftrightarrow \exists v(t \subseteq v \wedge \mathrm{Trans}(v) \wedge \mathrm{Fin}(v))$, i.e., t is hereditarily finite,
- $\mathrm{Ctbl}(t) \leftrightarrow \exists f(\mathrm{Inj}(f) \wedge \mathrm{dom}(f) = t \wedge \forall y(y \in \mathrm{rng}(f) \to \mathrm{FinOrd}(y)))$, i.e., t is countable,
- $\mathrm{HCtbl}(t) \leftrightarrow \exists v(t \subseteq v \wedge \mathrm{Trans}(v) \wedge \mathrm{Ctbl}\, v)$, i.e., t is hereditarily countable.

Here, f, r, s, t range over terms. Working in BS^0, note that r is a relation iff

$$(\forall x \in r)(x \text{ is an ordered pair } \langle u, v\rangle \text{ with objects } u, v).$$

It is straightforward to show that the above assertion is Δ_0. A relation r is said to be regular if

$$\forall u(\mathrm{S}(u) \wedge u \neq \emptyset \to (\exists x \in u)(\forall y \in u)(\langle y, x\rangle \notin r)).$$

Intuitively, a relation r is regular iff every non-empty subset of its field has an r-minimal element. Finally, we can turn to defining the additional axioms and corresponding theories.

Axiom Beta: For every regular relation r, there exists a collapsing function f, i.e., f is a function with $\mathrm{dom}(f) = \mathrm{field}(r)$ and

$$f'x = f''\{y : \langle y, x\rangle \in r\} \tag{Beta}$$

for all $x \in \mathrm{field}(r)$.

Axiom of Countability:

$$\forall u(\mathrm{S}(u) \to u \text{ is hereditarily countable}). \tag{C}$$

Axiom of Infinity:

$$\exists u(\mathrm{S}(u) \wedge \emptyset \in u \wedge (\forall x, y \in u)(x \cup \{y\} \in u)). \tag{Inf}$$

Definition 7.2. BETA_0 is the formal system in the language $\mathcal{L}_{\mathrm{s}}$ consisting of BS^2 together with (Beta). The system BETA is obtained from BS by adding (Beta). Equivalently, BETA matches BETA_0 plus ($\mathcal{L}_{\mathrm{s}}$-$\mathrm{I}_\in$) and ($\mathcal{L}_{\mathrm{s}}$-I_{N}).

Note that our object coding via representation trees is not sufficient any more, as the existence of ω is provable in BETA_0. However, this does not create a problem as we can relate BETA_0 to the well-known set theory $\mathsf{ATR}_0^{\mathsf{set}}$ by Simpson.

Lemma 7.3. *Working in* BETA_0 *the existence of* ω*, i.e., the first limit ordinal, can be proved. Moreover, there exists functions corresponding to the usual addition and multiplication on* ω*.*

Proof. By (O3) and (RC), the class $\mathrm{N} \times \mathrm{N}$ exists properly as set. By (Δ_0-Sep), there exists a set r such that

$$r = \{\langle m, n\rangle : m, n \in \mathrm{N} \wedge m < n\},$$

where $<$ is the usual p.r. less relation on the natural numbers. Clearly, r is a regular relation since every subset of N has a minimal element w.r.t. $<$. This is an immediate consequence of (Δ_0-I_{N}). According to Axiom Beta, r has a collapsing function f. Put $\omega := \mathrm{rng}(f)$. By (RC), ω exists properly. It remains to show that ω has the desired properties. For transitivity, consider x, y such that $x \in \omega$ and $y \in x$. By definition, let $m \in \mathrm{N}$ such that $x = f'm$. By construction of f and since $y \in x$ we can infer that $y = f'n$ for some $n \in \mathrm{N}$ with $n < m$. It follows that $y \in \omega$. Following a similar pattern, it is straightforward to show that ω is linearly ordered by $\in$ as this property is inherited from $<$. This proves that ω is an ordinal.

We content ourselves with proving the existence of a function corresponding to addition. For multiplication the strategy is the same. For notational simplicity we work with p.r. function symbols. Strictly speaking, we only have symbols for p.r. relations at our disposal, but since graphs of p.r. function are p.r., this abuse of notation does not pose a problem. By (RC) the class $\omega^3 = (\omega \times \omega) \times \omega$ exists properly. By (Δ_0-Sep) we can define addition as the set of all $\langle\langle u, v\rangle, w\rangle \in \omega^3$ such that

$$(\exists m, n, k \in N)(u = f(m) \wedge v = f(n) \wedge w = f(k) \wedge k = m + n),$$

with the collapsing function f of r serving as parameter. This concludes the proof. $\square$

7.2. Relation to $\mathsf{ATR}_0^{\mathsf{set}}$ and $\mathsf{ATR}^{\mathsf{set}}$

Simpson introduced a set-theoretic counterpart to ATR_0, denoted $\mathsf{ATR}_0^{\mathsf{set}}$, cf. [Sim09], VII.3. $\mathsf{ATR}_0^{\mathsf{set}}$ is formulated in a set-theoretic language without urelements, which can be viewed as a sublanguage of our language $\mathcal{L}_{\mathrm{s}}$. Keeping this in mind, the following theorem establishes a precise relation between BETA_0 and $\mathsf{ATR}_0^{\mathsf{set}}$.

Theorem 7.4. *We have the following inclusions:*

(a) BETA_0 *is a subsystem of* $\mathsf{ATR}_0^{\mathsf{set}}$.

(b) $\mathsf{ATR}_0^{\mathsf{set}} \setminus (\mathrm{C})$ *is a subsystem of* BETA_0.

Proof. (a): We work in $\mathsf{ATR}_0^{\mathsf{set}}$ and show that every axiom of BETA_0 is derivable. Referring to [Sim09], Lemma VII.3.7 and Theorem VII.3.9, it is clear that N can be represented by ω. Moreover, the constants $\underline{n}$ for $n \in \mathbb{N}$, and all p.r. relation symbols can be faithfully represented on ω. We content ourselves with deriving (Δ_0-I_{N}) and (Δ_0-$\mathrm{I}_{\in}$). All remaining axioms of BETA_0 are clearly valid in $\mathsf{ATR}_0^{\mathsf{set}}$. Since ω plays the role of N, (Δ_0-I_{N}) is an immediate consequence of (Δ_0-$\mathrm{I}_{\in}$). Thus, it suffices to derive (Δ_0-$\mathrm{I}_{\in}$). To this end, consider the Δ_0 formula $\mathscr{A}(x)$ and suppose that u is a set satisfying $\mathscr{A}(u)$. By (C), let v be transitive such that $u \subseteq v$. Using (Δ_0-Sep), let $w = \{x \in v \cup \{u\} : \mathscr{A}(x)\}$. Note that $v \cup \{u\}$ is transitive. By assumption, $\mathscr{A}(u)$ holds, therefore $w \neq \emptyset$. Using (Reg), let $z \in w$ such that $(\forall y \in z)(y \notin w)$. Thus, we have $\mathscr{A}(z)$ and, by transitivity of $v \cup \{u\}$, $(\forall y \in z)\neg\mathscr{A}(y)$. This establishes ($\Delta_0$-$\mathrm{I}_{\in}$), concluding the proof of (a).
(b): We work in BETA_0 and only consider the Axiom of Regularity and the Axiom of Infinity. All other axioms of $\mathsf{ATR}_0 \setminus (\mathrm{C})$ are clearly valid in BETA_0. For the Axiom of Regularity, let $v \neq \emptyset$. Taking the contraposition

of (Δ_0-$\mathrm{I}_\in$) applied to the formula $x \notin v$ and using that $v \neq \emptyset$, there exists $x \in v$ such that

$$(\forall y \in x)(y \notin v).$$

This proves the Axiom of Regularity. For the Axiom of Infinity we have to show that there exists a set u such that

$$\emptyset \in u \wedge (\forall x, y \in u)(x \cup \{y\} \in u).$$

Using Ax(PA), let E be the p.r. relation symbol on N such that nEm iff there exists $m_1 > m_2 > \ldots > m_j$ with

$$m = 2^{m_1} + \ldots + 2^{m_j}$$

and $n = m_i$ for some $1 \leq i \leq j$. By (Δ_0-I_{N}), the relation E is regular and $\mathrm{field}(E) = \mathrm{N}$. Thus, Axiom Beta yields a function f such that $\mathrm{dom}(f) = \mathrm{N}$ and for all $n \in \mathrm{N}$,

$$f'n = f'' \{m : mEn\}.$$

Put $u = \mathrm{rng}(f)$. We show that u validates the Axiom of Infinity. First, observe that

$$f(0) = f'' \{m : mE0\} = f''\emptyset = \emptyset.$$

Thus, $\emptyset \in u$. Next, suppose $x, y \in u$, i.e., there exists $k, l \in \mathrm{N}$ such that

$$x = f(k) \wedge y = f(l).$$

We let

$$h = \begin{cases} k & lEk, \\ k + \underline{2}^l & \text{otherwise.} \end{cases}$$

It follows that $f(h) = x \cup \{y\}$. Observe that $h = k + \underline{2}^l$ for $h \in \mathrm{N}$ is obviously expressible in BETA_0. Hence, the above abuse of notation does not pose a problem. This establishes (b). □

By the above theorem, BETA_0 + (C) is equivalent to $\mathsf{ATR}_0^{\mathsf{set}}$. In that sense, urelements become superfluous and it does not matter whether we use urelemens or ordinals to represent natural numbers. As mentioned earlier, our notion of representation tree is inappropriate for representing Axiom Beta. This can be overcome by making use of suitable trees as introduced in [Sim09], Definition VII.3.10.

Definition 7.5. Given any formula $\mathscr{A}$ of $\mathcal{L}_s$, we use $\|\mathscr{A}\|_2$ to denote its translation into $\mathcal{L}_2$ as introduced in [Sim09], Definition VII.3.15.

Following [Sim09], we now have all means to link ATR_0 and BETA_0. For completeness we cover all details.

Lemma 7.6. *Within* BS^2*, the class of ordinals is linearly ordered by* $\in$*.*

Proof. Irreflexivity is a direct consequence of (Reg) and (Pair). Transitivity follows since ordinals are transitive. For totality, consider $\alpha \neq \beta$. Without loss of generality, $\alpha \setminus \beta \neq \emptyset$. By (Reg), let $x^* \in \alpha \setminus \beta$ such that

$$(\forall y \in \alpha \setminus \beta)(y \notin x^*). \tag{7.1}$$

By (7.1) and Trans(α) we obtain $x^* \subseteq \alpha \cap \beta$. Moreover, $\alpha \cap \beta \subseteq x^*$ since otherwise $x^* \in \beta$ by totality of $\in\restriction\alpha$ and since β is transitive. If $\beta \subseteq \alpha$ we are done. Suppose this is not the case, then arguing as above yields a set $y^* \in \beta$ with $y^* = \alpha \cap \beta$, hence $y^* = x^*$. But then $x^* \in \beta$, contradicting $x^* \notin \beta$. Thus, we must have $\beta \subseteq \alpha$. It follows that $\beta = x^* \in \alpha$. □

Definition 7.7. Working in ACA_0, let X, Y be well orderings. We write $f : |X| = |Y|$ to indicate that f is a function $f\colon \text{field}(X) \to \text{field}(Y)$ such that $(\forall k \in \text{field}(Y))(\exists i \in \text{field}(X))(f(i) = k)$ and $\forall i, j(i \leq_X j \leftrightarrow f(i) \leq_Y f(j))$, i.e., f is an order-preserving bijection from X to Y. We write $f : |X| < |Y|$ if $f : |X| = |\,\{l \in \text{field}(Y) : l <_Y k\}\,|$ for some $k \in \text{field}(Y)$, and $f : |X| > |Y|$ if $f : |\,\{i \in \text{field}(X) : i <_X j\}\,| = |Y|$ for some $j \in \text{field}(X)$. Moreover, $|X| = |Y|$ if there exists some f such that $f : |X| = |Y|$, and $|X| < |Y|$ if $f : |X| = |Y|$ for some f. Additional notations such as $|X| \leq |Y|$, $|X| > |Y|$, $f : |X| \geq |Y|$ and so on, are to be understood accordingly.

Theorem 7.8. *Over* ACA_0*, the schema* (ATR) *is equivalent to the comparability of well orderings,* CWO *for short, i.e.,*

$$\forall V, W(\text{WO}(V) \wedge \text{WO}(W) \to |V| \leq |W| \vee |V| \geq |W|).$$

Proof. See [Sim09], Theorem V.6.8. □

Theorem 7.9. *We have the following interpretability results.*

(a) Every axiom of ATR_0 *is a theorem of* BETA_0*.*

(b) If $\mathscr{A}$ *is an axiom of* BETA_0*, then* $\|\mathscr{A}\|_2$ *is a theorem of* ATR_0*.*

Proof. (a): Recall that we can view $\mathcal{L}_2$ as a sublanguage of $\mathcal{L}_{\mathrm{s}}$. By Theorem 4.4 we then have $\mathsf{ACA}_0 \subseteq \mathsf{BS}^0$. Hence, it suffices to show that all instances of (ATR) are provable in BETA_0. Referring to Theorem 7.8 we will instead derive CWO. We closely follow [Sim09]. Working in BETA_0, assume that $x, y \subseteq \mathrm{N}$ are well orderings, cf. section 1.4. Put $r_x = \{\langle n, m\rangle : n <_x m\}$. Since x is a well ordering, the relation r_x is regular. By Axiom Beta, let f_x be the collapsing function of r_x. Put $\alpha_x = \mathrm{rng}(f_x)$. One can easily verify that α_x is an ordinal. Moreover, $f_x : |\,\mathrm{field}(x)| = |\alpha_x|$, i.e., f_x is an order-preserving bijection from field(x) to α_x. Analogously, we can define r_y, f_y and α_y. By Lemma 7.6 we have $\alpha_x \subseteq \alpha_y$ or $\alpha_y \subseteq \alpha_x$. If $\alpha_x \subseteq \alpha_y$ we can set

$$g = \left\{\langle m, n\rangle : m \in \mathrm{field}(x) \wedge n \in \mathrm{field}(y) \wedge f'_x m = f'_y n\right\}.$$

This defines a proper function since f_x and f_y are uniquely determined by $(\Delta_0\text{-}\mathrm{I}_\in)$. It follows that $g : |x| \leq |y|$. Similarly, $\alpha_y \subseteq \alpha_x$ implies $|x| \geq |y|$. This shows CWO, establishing (a).
(b): Let $\mathscr{A}$ be an axiom of BETA_0. Using Theorem 7.4(a), $\mathscr{A}$ is a theorem of $\mathsf{ATR}_0^{\mathsf{set}}$. Referring to [Sim09], Lemma VII.3.20 the claim follows. □

This theorem can be easily extended to BETA. To achieve this we have to replace (ATR) by (BI) on the side of second order arithmetic.

Definition 7.10. Given an $\mathcal{L}_2$ formula $\mathscr{A}(i)$, let $\mathrm{TI}(\mathscr{A}, X)$ be the formula

$$\forall j((\forall i <_X j)\mathscr{A}(i) \to \mathscr{A}(j)) \to \forall j \mathscr{A}(j).$$

The schema (BI) of bar induction consists of all formulas

$$\forall X(\mathrm{WF}(X) \to \mathrm{TI}(X, \mathscr{A})),$$

where $\mathscr{A}$ ranges over all $\mathcal{L}_2$ formulas. For the definition of $\mathrm{WF}(X)$ we refer to section 1.4.

Theorem 7.11. *The following assertions hold.*

(a) Every axiom of $\mathsf{ACA}_0 + (\mathrm{BI})$ *is a theorem of* BETA.

(b) If $\mathscr{A}$ *is an axiom of* BETA*, then* $\|\mathscr{A}\|_2$ *is a theorem of* $\mathsf{ACA}_0 + (\mathrm{BI})$.

Proof. It is a classic result that all instances of (BI) can be proved by means of (Beta) and $(\mathcal{L}_{\mathrm{s}}\text{-}\mathrm{I}_\in)$, cf. [Jäg86]. Together with Theorem 4.4(a) this shows (a). For (b) we use Theorem 7.4(a) and refer to [Sim09], Theorem VII.3.34 and Exercise VII.3.38. □

We continue by stating the so-called quantifier theorem, which links formulas of $\mathcal{L}_2$ with formulas of $\mathcal{L}_\mathrm{s}$. The theorem is based on [Sim09], Theorem VII.3.24. The first assertion can also be obtained from corresponding results in [Jäg79, Jäg86].

Theorem 7.12. *For any natural number $n \in \mathbb{N}$ we have the following:*

(a) Each Σ^1_{n+2} formula of $\mathcal{L}_2$ is equivalent – provably in BETA_0 – to a Σ_{n+1} formula of $\mathcal{L}_\mathrm{s}$.

(b) If $\mathscr{A}$ is a Σ_n formula of $\mathcal{L}_\mathrm{s}$, then $\|\mathscr{A}\|_2$ is equivalent – provably in ATR_0 – to a Σ^1_{n+1} formula of $\mathcal{L}_2$.

(c) If $\mathscr{A}$ is a Σ formula of $\mathcal{L}_\mathrm{s}$, then $\|\mathscr{A}\|_2$ is equivalent – provably in $\Delta^1_2\text{-}\mathsf{CA}_0$ – to a Σ^1_2 formula of $\mathcal{L}_2$.

Proof. (a) and (b) follow as in [Sim09], Theorem VII.3.24. For (c) we work in $\Delta^1_2\text{-}\mathsf{CA}_0$ and proceed by induction on the build-up of $\mathscr{A}$. If $\mathscr{A}$ is an atomic formula or built-up using connectives, we can proceed as in the proof of [Sim09], Theorem VII.3.24. item 1. Next, suppose that $\mathscr{A}$ is of the form $(\forall v_i \in v_j)\mathscr{B}$. By the induction hypothesis $\|\mathscr{B}\|_2$ is equivalent to a formula of the form $\exists X \forall Y \mathscr{C}(X, Y, V_i, V_j)$ with $\mathscr{C}(X, Y, V_i, V_j)$ being arithmetical. By [Sim09], Lemma VII.3.17. $\|\mathscr{A}\|_2$ is equivalent to the formula

$$\forall n(\langle n\rangle \in V_j \rightarrow \exists V_i(V_i =^* V_j^{\langle n\rangle} \wedge \exists X \forall Y \mathscr{C}(X, Y, V_i, V_j))).$$

Pulling out quantifiers, $\|\mathscr{A}\|_2$ is equivalent to the formula

$$\forall n \exists V_i \exists X \forall Y(\langle n\rangle \in V_j \rightarrow V_i =^* V_j^{\langle n\rangle} \wedge \mathscr{C}(X, Y, V_i, V_j)).$$

Note that $V_i =^* V_j^{\langle n\rangle}$ contains an existential quantifier as it is of the form

$$\exists Z(\mathrm{Iso}(Z, V_i \oplus V_j^{\langle n\rangle}) \wedge \langle\langle 0\rangle, \langle 1\rangle\rangle \in Z).$$

By some further manipulations $\|\mathscr{A}\|_2$ is equivalent to

$$\begin{aligned}&\forall n \exists V_i \exists X \forall Y (\langle n \rangle \in V_j \to \\ &\qquad \mathrm{Iso}((X)_0, V_i \oplus V_j^{\langle n \rangle}) \wedge \langle \langle 0 \rangle, \langle 1 \rangle \rangle \in (X)_0 \wedge \mathscr{C}((X)_1, Y, V_i, V_j)).\end{aligned}$$

By [Sim09], Theorem VII.6.9. the schema (Σ^1_2-AC) is available in $\mathsf{\Delta}^1_2\text{-}\mathsf{CA}_0$. Thus, $\|\mathscr{A}\|_2$ is equivalent to

$$\begin{aligned}&\exists W \forall n \exists X \forall Y (\langle n \rangle \in V_j \to \\ &\qquad \mathrm{Iso}((X)_0, (W)_n \oplus V_j^{\langle n \rangle}) \wedge \langle \langle 0 \rangle, \langle 1 \rangle \rangle \in (X)_0 \wedge \mathscr{C}((X)_1, Y, (W)_n, V_j)).\end{aligned}$$

It is now easy to see that $\|\mathscr{A}\|_2$ is equivalent to a Σ^1_2 formula. More precisely, the $\forall n$ quantifier in the above formula can be pulled inwards by applying (Σ^1_2-AC) once more. The remaining existential set quantifiers $\exists W$ and $\exists X$ can be combined as above using the pairing function. This concludes the case. The remaining cases, i.e., $\mathscr{A}$ being of the form $\exists v_i \mathscr{B}$ or $(\exists v_i \in v_j)\mathscr{B}$ follow in a similar manner. This shows (c). $\square$

Recall that we use $|\,\mathsf{T}\,|$ to denote the proof-theoretic ordinal of a given formal system T. We end the section by mentioning several proof-theoretic results.

Theorem 7.13. *We have the following results regarding proof-theoretic strength.*

(a) $|\mathsf{BETA}_0| = |\mathsf{ATR}_0| = \Gamma_0$.

(b) $|\mathsf{ATR}| = \Gamma_{\varepsilon_0}$.

(c) $|\mathsf{BETA}| = |\mathsf{ACA}_0 + (\mathrm{BI})| = \Psi(\varepsilon_{\Omega+1})$. *(Bachmann-Howard ordinal)*

Proof. (a) follows from Theorem 7.9 and the well-known fact that Γ_0 is the proof-theoretic ordinal of ATR_0. For (b) we refer to [JKSS99, JS00]. Finally, (c) follows from Theorem 7.11 and standard proof-theoretic results that tell us that $\Psi(\varepsilon_{\Omega+1})$, i.e., the Bachmann-Howard ordinal, is the proof-theoretic ordinal of $\mathsf{ACA}_0 + (\mathrm{BI})$. $\square$

7.3. Adding Σ and Π reduction

The goal of this section is to discuss the effect of adding reduction principles to BETA_0 and related systems. As in chapter 6 we are interested in (Π-Red) and (Σ-Red), i.e., Σ and Π reduction. We start by listing some additional axiom schemas of $\mathcal{L}_2$ and $\mathcal{L}_\mathrm{s}$ that are crucial in this context.

Comprehension schemas (Δ^1_2-CA) **and** (Π^1_2-CA)

For all Σ^1_2 formulas $\mathscr{A}(n)$ and all Π^1_2 formulas $\mathscr{B}(n)$ of $\mathcal{L}_2$,

$$\forall n(\mathscr{A}(n) \leftrightarrow \mathscr{B}(n)) \to \exists X \forall n(n \in X \leftrightarrow \mathscr{A}(x)), \tag{Δ^1_2-CA}$$

where X must not occur freely in $\mathscr{A}(n)$.

For all Π^1_2 formulas $\mathscr{A}(n)$ of $\mathcal{L}_2$,

$$\exists X \forall n(n \in X \leftrightarrow \mathscr{A}(n)), \tag{Π^1_2-CA}$$

with X not occurring freely in $\mathscr{A}(n)$.

Π^1_2 and Σ^1_2 reduction

For all Σ^1_2 formulas $\mathscr{A}(n)$ and all Π^1_2 formulas $\mathscr{B}(n)$ of $\mathcal{L}_2$, the schema (Π^1_2-Red) consists of all formulas of the form

$$\begin{aligned} &\forall n(\mathscr{A}(n) \to \mathscr{B}(n)) \to \\ &\qquad \exists Y(\forall n(\mathscr{A}(n) \to n \in Y) \land \forall n(n \in Y \to \mathscr{B}(n))). \end{aligned} \tag{Π^1_2-Red}$$

Analogously, the schema (Σ^1_2-Red) contains exactly all formulas of the form

$$\begin{aligned} &\forall n(\mathscr{B}(n) \to \mathscr{A}(n)) \to \\ &\qquad \exists Y(\forall n(\mathscr{B}(n) \to n \in Y) \land \forall n(n \in Y \to \mathscr{A}(n))), \end{aligned} \tag{Σ^1_2-Red}$$

with $\mathcal{L}_2$ formulas $\mathscr{A}(n)$ and $\mathscr{B}(n)$ as above.

As before, we write $\mathsf{\Delta^1_2\text{-}CA}$ for the theory $\mathsf{ACA_0} + (\Delta^1_2\text{-CA})$, and so on, cf. section 2.2. To deal with Π^1_2 and Σ^1_2 reduction we use additional results.

Theorem 7.14 (Buchholz-Schütte, Simpson)**.**

(a) $\mathsf{ACA_0} + (\Sigma^1_2\text{-Red})$ *is equivalent to* $\mathsf{\Delta^1_2\text{-}CA_0}$.

(b) $\mathsf{ACA_0} + (\Pi^1_2\text{-Red})$ *is equivalent to* $\mathsf{\Pi^1_2\text{-}CA_0}$.

Proof. A proof for (a) can be found in [BS88]; (b) is mentioned in [Sim09], Exercise VII.6.14. $\square$

Recall that by Theorem 2.2, (Π^1_1-Red) is equivalent to (ATR) over ACA_0. Moreover, we have the following.

Theorem 7.15 (Simpson)**.** *The theory* ACA_0 + (BI) *proves all instances of* (ATR).

Proof. The assertion is an immediate consequence of [Sim09], Corollary VII.2.19. $\square$

By Theorem 7.9 and Theorem 7.11 combined with Theorem 7.12(a) we can infer

$$\begin{aligned}
\mathsf{ATR}_0 + (\Sigma^1_2\text{-Red}) &\subseteq \mathsf{BETA}_0 + (\Sigma\text{-Red}),\\
\mathsf{ACA}_0 + (\mathrm{BI}) + (\Sigma^1_2\text{-Red}) &\subseteq \mathsf{BETA} + (\Sigma\text{-Red}),\\
\mathsf{ATR}_0 + (\Pi^1_2\text{-Red}) &\subseteq \mathsf{BETA}_0 + (\Pi\text{-Red}),\\
\mathsf{ACA}_0 + (\mathrm{BI}) + (\Pi^1_2\text{-Red}) &\subseteq \mathsf{BETA} + (\Pi\text{-Red}).
\end{aligned}$$

Together with Theorem 7.14 we can estimate the effect of adding (Σ-Red) and (Π-Red) to, respectively, BETA_0 and BETA from above.

Theorem 7.16. *We have the following inclusions.*

(a) $\Delta^1_2\text{-}\mathsf{CA}_0 \subseteq \mathsf{ACA}_0 + (\Sigma^1_2\text{-Red}) \subseteq \mathsf{BETA}_0 + (\Sigma\text{-Red})$.

(b) $\Delta^1_2\text{-}\mathsf{CA}_0 + (\mathrm{BI}) \subseteq \mathsf{ACA}_0 + (\Sigma^1_2\text{-Red}) + (\mathrm{BI}) \subseteq \mathsf{BETA} + (\Sigma\text{-Red})$.

(c) $\Pi^1_2\text{-}\mathsf{CA}_0 \subseteq \mathsf{ACA}_0 + (\Pi^1_2\text{-Red}) \subseteq \mathsf{BETA}_0 + (\Pi\text{-Red})$.

(d) $\Pi^1_2\text{-}\mathsf{CA}_0 + (\mathrm{BI}) \subseteq \mathsf{ACA}_0 + (\Pi^1_2\text{-Red}) + (\mathrm{BI}) \subseteq \mathsf{BETA} + (\Pi\text{-Red})$.

To show that these bounds are sharp w.r.t. proof-theoretic strength, we first observe by Theorem 7.12(c) and Theorem 7.14 that for every (closed) instance $\mathscr{A}$ of (Σ-Red) and (Π-Red), the corresponding $\mathcal{L}_2$ formula $\|\mathscr{A}\|_2$ is derivable in, respectively, $\mathsf{ATR}_0 + (\Sigma^1_2\text{-Red})$ and $\mathsf{ATR}_0 + (\Pi^1_2\text{-Red})$. Therefore, by Theorem 7.9 and Theorem 7.11 we obtain the following for all sentences $\mathscr{A}$ of $\mathcal{L}_s$:

$$\begin{array}{lcl}
\mathsf{BETA}_0 + (\Sigma\text{-Red}) \vdash \mathscr{A} & \Longrightarrow & \mathsf{ATR}_0 + (\Sigma^1_2\text{-Red}) \vdash \|\mathscr{A}\|_2, \\
\mathsf{BETA} + (\Sigma\text{-Red}) \vdash \mathscr{A} & \Longrightarrow & \mathsf{ACA}_0 + (\Sigma^1_2\text{-Red}) + (\mathrm{BI}) \vdash \|\mathscr{A}\|_2, \\
\mathsf{BETA}_0 + (\Pi\text{-Red}) \vdash \mathscr{A} & \Longrightarrow & \mathsf{ATR}_0 + (\Pi^1_2\text{-Red}) \vdash \|\mathscr{A}\|_2, \\
\mathsf{BETA} + (\Pi\text{-Red}) \vdash \mathscr{A} & \Longrightarrow & \mathsf{ACA}_0 + (\Pi^1_2\text{-Red}) + (\mathrm{BI}) \vdash \|\mathscr{A}\|_2.
\end{array}$$

Using Theorem 7.14 once more we immediately get the following result.

Theorem 7.17. *The following hold for every sentence $\mathscr{A}$ of $\mathcal{L}_s$:*

(a) $\mathsf{BETA}_0 + (\Sigma\text{-Red}) \vdash \mathscr{A} \quad \Longrightarrow \quad \Delta^1_2\text{-}\mathsf{CA}_0 \vdash \|\mathscr{A}\|_2$.

(b) $\mathsf{BETA} + (\Sigma\text{-Red}) \vdash \mathscr{A} \quad \Longrightarrow \quad \Delta^1_2\text{-}\mathsf{CA}_0 + (\mathrm{BI}) \vdash \|\mathscr{A}\|_2$.

(c) $\mathsf{BETA}_0 + (\Pi\text{-Red}) \vdash \mathscr{A} \quad \Longrightarrow \quad \Pi^1_2\text{-}\mathsf{CA}_0 \vdash \|\mathscr{A}\|_2$.

(d) $\mathsf{BETA} + (\Pi\text{-Red}) \vdash \mathscr{A} \quad \Longrightarrow \quad \Pi^1_2\text{-}\mathsf{CA}_0 + (\mathrm{BI}) \vdash \|\mathscr{A}\|_2$.

Referring to [Sim09], Lemma VII.3.19., we combine Theorem 7.16 and Theorem 7.17 to determine the proof-theoretic strength of (Σ-Red) and (Π-Red) added to BETA_0 and BETA.

Corollary 7.18. *The following proof-theoretic equivalences hold:*

$$\begin{array}{rcl}
|\mathsf{BETA}_0 + (\Sigma\text{-Red})| & = & |\Delta^1_2\text{-}\mathsf{CA}_0|, \\
|\mathsf{BETA} + (\Sigma\text{-Red})| & = & |\Delta^1_2\text{-}\mathsf{CA}_0 + (\mathrm{BI})|, \\
|\mathsf{BETA}_0 + (\Pi\text{-Red})| & = & |\Pi^1_2\text{-}\mathsf{CA}_0|, \\
|\mathsf{BETA} + (\Pi\text{-Red})| & = & |\Pi^1_2\text{-}\mathsf{CA}_0 + (\mathrm{BI})|.
\end{array}$$

This clarifies the picture for (Σ-Red) and (Π-Red) in the context of BETA_0 and its extensions. The situation becomes completely different when we move to Kripke-Platek set theory. The next section will provide a short oversight, whereby we refer to further work which is to be published.

7.4. Moving to Kripke-Platek set theory

By adding collection for Δ_0 formulas we basically arrive in the realm of Kripke-Platek set theory. More precisely, up to equivalence, Kripke-Platek set theory KP can obtained from BS by adding (Inf) and the schema of Δ_0 collection, i.e.,

$$(\forall x \in a)\exists y \mathscr{A}(x,y) \to \exists z(\forall x \in a)(\exists y \in z)\mathscr{A}(x,y)$$

for all Δ_0 formulas $\mathscr{A}(x,y)$. Note that BS includes the natural numbers as urelements. However, these are not needed anymore when adding (Inf). Similarly, we obtain KP_0, i.e., KP with regularity restricted to sets, as

$$\mathsf{KP}_0 \equiv \mathsf{BS}^2 + (\text{Inf}) + (\Delta_0\text{-Col}).$$

Thus, recalling section 7.3, KP_0 + (Beta) and KP + (Beta) are equivalent to the theories $\mathsf{ATR}_0^{\mathsf{set}} \setminus (\text{C}) + (\Delta_0\text{-Col})$ and $\mathsf{ATR}^{\mathsf{set}} \setminus (\text{C}) + (\Delta_0\text{-Col})$, respectively. However, when Axiom Beta is missing, the situation turns out to be very different. This is because $\mathsf{ATR}_0^{\mathsf{set}}$ and KP are not compatible. More precisely, it can be shown that

$$\mathsf{KP} \subsetneq \mathsf{ATR}^{\mathsf{set}} \quad \text{and} \quad \mathsf{ATR}_0^{\mathsf{set}} \subsetneq \mathsf{KP}.$$

It is a very natural question in our context to determine the effect of adding reduction principles to KP and KP_0 as in previous sections. First answers to such questions and related open problems are discussed in [BJar].

A. Appendix

ATR_0 without field variable

In section 2.1 the system ATR_0 featuring arithmetical transfinite recursion was introduced. The definition includes a so-called field variable. It turns out that this variable can be omitted without loss of generality. This is also true for variants of ATR_0 without set parameters. The corresponding systems and equivalence proofs shall be presented next.

For any arithmetical formula $\mathscr{A}(n, X)$, let $\mathcal{H}^{\circ}_{\mathscr{A}}(W, Y)$ be the formula asserting that $\mathrm{LO}(W)$ and

$$Y = \left\{ \langle n, j \rangle : j \in \mathrm{field}(W) \wedge \mathscr{A}(n, (Y)^{Wj}) \right\}.$$

In contrast to $\mathcal{H}_{\mathscr{A}}(W, Y)$ as introduced in section 2.1, the field variable j is not allowed to occur in the formula $\mathscr{A}(n, X)$ over which is iterated. Of course, additional parameters of $\mathscr{A}(n, X)$ also occur in $\mathcal{H}^{\circ}_{\mathscr{A}}(W, Y)$.

Arithmetical transfinite recursion without field variable

The axiom schema (oATR) consists of all formulas

$$\forall W(\mathrm{WO}(W) \to \exists Y \mathcal{H}^{\circ}_{\mathscr{A}}(W, Y)),$$

where $\mathscr{A}(n, X)$ is arithmetical. The corresponding systems extending, respectively, ACA_0 and ACA, with (oATR) are denoted oATR_0 and oATR. (oATR) is defined in compliance with the original schema of arithmetic transfinite recursion as defined in [Sim09]. However, several applications of arithmetic transfinite recursion in [Sim09] seem to rely on the field variable. With the considerations presented here, we aim to clarify the situation.

Arithmetical transfinite recursion w/o set-parameters and field variable

The axiom schema (oATR^-) consists of all formulas

$$\forall W(\mathrm{WO}(W) \to \exists Y \mathcal{H}^{\circ}_{\mathscr{A}}(W, Y)),$$

where $\mathscr{A}(n,X)$ is arithmetical and X is the only set variable allowed to occur freely in $\mathscr{A}(n,X)$. The corresponding systems extending, respectively, ACA_0 and ACA, with (oATR$^-$) are denoted oATR_0^- and oATR^-.

Theorem A.1. *The systems* ATR_0 *and* oATR_0 *are equivalent.*

Proof. Clearly, $\mathsf{oATR}_0 \subseteq \mathsf{ATR}_0$ since the schema (ATR) is more general than (oATR). For the converse direction, suppose we are working in oATR_0. Let W be a well ordering and consider an arithmetical formula $\mathscr{A}(n,j,X)$. Our goal is to prove the existence of a set Z satisfying $\mathcal{H}_{\mathscr{A}}(W,Z)$. To achieve this we define the formula

$$\begin{aligned}\mathscr{A}^*(n,X,W) :\equiv \exists k,m,w\big(n = \langle m,w\rangle \wedge \mathrm{Hier}_W(k,X) \wedge \\ \big((w=0 \wedge \neg\mathscr{A}(m,k,X^{-1})) \vee \\ (w=1 \wedge \mathscr{A}(m,k,X^{-1}))\big)\big)\end{aligned}$$

where

$$\begin{aligned}X^{-1} &:= \{\langle m,j\rangle : \langle\langle m,1\rangle,j\rangle \in X\}\,,\\ \mathrm{Hier_W}(k,X) &:\equiv \big(X=\emptyset \wedge k=\min(W)\big) \vee \big(X\neq\emptyset \wedge \\ &\qquad \forall x(x\in X \to \exists m,l(x=\langle m,l\rangle \wedge l <_W k)) \wedge \\ &\qquad \forall l(l <_W k \to \exists m(\langle m,l\rangle \in X))\big),\end{aligned}$$

and $\min(X)$ denoting the minimal element of X. Let Y be the result of applying (oATR) to $\mathscr{A}^*(n,X,W)$ (with W as additional parameter). Consider $j\in\mathrm{field}(W)$. By arithmetical transfinite induction along W we shall show that

$$(Y)_j \neq \emptyset \wedge \forall k(\mathrm{Hier}_W(k,(Y)^{Wj}) \leftrightarrow k=j).$$

In case $j=\min(W)$, then $(Y)^{Wj}=\emptyset$. The above then holds since we have for all k that $\mathrm{Hier}_W(k,\emptyset)$ iff $k=\min(W)$, and so

$$\begin{aligned}(Y)_{\min(W)} &= \{n : \mathscr{A}^*(n,\emptyset,W)\}\\ &= \{\langle m,0\rangle : \neg\mathscr{A}(m,\min(W),\emptyset)\} \cup \{\langle m,1\rangle : \mathscr{A}(m,\min(W),\emptyset)\}\\ &\neq \emptyset.\end{aligned}$$

Next, assume $j\neq\min(W)$. By the induction hypothesis and since $Y\subseteq$

$\mathrm{Nat} \times \mathrm{field}(W)$, it follows that

$$\forall l(((Y)^{Wj})_l \neq \emptyset \leftrightarrow l <_W j),$$

hence

$$\forall k(\mathrm{Hier}_W(k, (Y)^{Wj}) \leftrightarrow k = j). \tag{A.1}$$

By definition we have $n \in (Y)_j$ iff n is a pair $\langle m, w \rangle$ such that for some $k \in \mathrm{field}(W)$ it holds that $\mathrm{Hier}_W(k, (Y)^{Wj})$ and

$$(w = 0 \wedge \neg\mathscr{A}(m, k, ((Y)^{Wj})^{-1})) \vee (w = 1 \wedge \mathscr{A}(m, k, ((Y)^{Wj})^{-1})).$$

By (A.1), it follows that $\langle m, w \rangle \in (Y)_j$ iff

$$(w = 0 \wedge \neg\mathscr{A}(m, j, ((Y)^{Wj})^{-1})) \vee (w = 1 \wedge \mathscr{A}(m, j, ((Y)^{Wj})^{-1})).$$

Thus, $(Y)_j$ is non-empty. This concludes the induction.

It remains to check that Y^{-1} is a set of the desired form. A simple calculation shows that for $j \in \mathrm{field}(W)$:

$$\begin{aligned}
((Y)^{Wj})^{-1} &= \left\{\langle m, i \rangle : \langle\langle m, 1\rangle, i\rangle \in (Y)^{Wj}\right\} \\
&= \{\langle m, i \rangle : \langle\langle m, 1\rangle, i\rangle \in Y \wedge i <_W j\} \\
&= \left\{\langle m, i \rangle : \langle m, i\rangle \in Y^{-1} \wedge i <_W j\right\} \\
&= (Y^{-1})^{Wj}.
\end{aligned}$$

By what we showed in the above induction, it follows that

$$\begin{aligned}
(Y^{-1})_j &= \left\{m : \langle m, j\rangle \in Y^{-1}\right\} \\
&= \{m : \langle\langle m, 1\rangle, j\rangle \in Y\} \\
&= \{m : \langle m, 1\rangle \in (Y)_j\} \\
&= \left\{m : \mathscr{A}(m, j, ((Y)^{Wj})^{-1})\right\} \\
&= \left\{m : \mathscr{A}(m, j, (Y^{-1})^{Wj})\right\},
\end{aligned}$$

hence Y^{-1} is an appropriate hierarchy. $\square$

Theorem A.2. *The systems* ATR_0^- *and* oATR_0^- *are equivalent.*

Before turning to the prove of the above theorem, we need two technical lemmas. These shall be presented and proved next.

Lemma A.3. *Working in* ACA_0*, consider a formula* $\mathscr{A}(n, j, X)$*, a well ordering* W *and a set* Y*. Setting*

$$\mathscr{A}^*(n, j, X) :\equiv n = 0 \vee \exists m(n = m + 1 \wedge \mathscr{A}(m, j, X^*))$$

where $X^* := \{\langle n, j\rangle : (n + 1, j) \in X\}$*, it follows that*

$$\mathcal{H}_{\mathscr{A}^*}(W, Y) \to \mathcal{H}_{\mathscr{A}}(W, Y^*).$$

Proof. Suppose $\mathcal{H}_{\mathscr{A}^*}(W, Y)$. $Y^* \subseteq \mathrm{Nat} \times \mathrm{field}(W)$ clearly exists as a set by (ACA). Note that $((Y)^{Wj})^* = (Y^*)^{Wj}$. For any number n and $j \in \mathrm{field}(W)$ we have

$$\begin{aligned} Y^*_j &= \{n : \langle n, j\rangle \in Y^*\} \\ &= \{n : \langle n + 1, j\rangle \in Y\} \\ &= \left\{n : \mathscr{A}^*(n + 1, j, (Y)^{Wj})\right\} \\ &= \left\{n : \mathscr{A}(n, j, ((Y)^{Wj})^*)\right\} \\ &= \left\{n : \mathscr{A}(n, j, (Y^*)^{Wj})\right\}. \end{aligned}$$

This establishes $\mathcal{H}_{\mathscr{A}}(W, Y^*)$. □

Lemma A.4. *Working in* ACA_0*, consider a formula* $\mathscr{A}(n, j, X)$*, a well ordering* W *and a set* Y*. Setting*

$$\begin{aligned} V &:= \{\langle 0, 0\rangle\} \cup \{\langle 0, i + 1\rangle : i \in \mathrm{field}(W)\} \cup \{\langle i + 1, j + 1\rangle : i \leq_W j\}, \\ Y^\bullet &:= \{\langle n, j\rangle : \langle n, j + 1\rangle \in Y\}, \end{aligned}$$

and $\mathscr{A}^\bullet(n, j, X) :\equiv \mathscr{A}(n, j - 1, X^\bullet)$*, it follows that*

$$\mathcal{H}_{\mathscr{A}^\bullet}(V, Y) \to \mathcal{H}_{\mathscr{A}}(W, Y^\bullet).$$

Proof. Assume $\mathcal{H}_{\mathscr{A}^\bullet}(V, Y)$. By (ACA), $Y^\bullet \subseteq \mathrm{Nat} \times \mathrm{field}(W)$ exists as a set. Our goal is to show $\mathcal{H}_{\mathscr{A}}(W, Y^\bullet)$. Since W is a well ordering, so is V. By definition of $\mathscr{A}^\bullet(n, j, X)$ we have for $j \in \mathrm{field}(W)$:

$$(Y)_{j+1} = \left\{n : \mathscr{A}(n, j, ((Y)^{V_{j+1}})^\bullet)\right\}.$$

It remains to check that $Y^\bullet$ iterates $\mathscr{A}(n, j, X)$ along W. Let n be any number and $j \in \mathrm{field}(W)$. We first observe that

$$\begin{aligned}((Y)^{V_{j+1}})^{\bullet} &= \left\{\langle n,i\rangle : \langle n,i+1\rangle \in (Y)^{Vj+1}\right\}\\ &= \{\langle n,i\rangle : \langle n,i+1\rangle \in Y \wedge i+1 <_V j+1\}\\ &= \{\langle n,i\rangle : \langle n,i+1\rangle \in Y \wedge i <_W j\}\\ &= \{\langle n,i\rangle : \langle n,i\rangle \in Y^{\bullet} \wedge i <_W j\}\\ &= (Y^{\bullet})^{Wj}.\end{aligned}$$

It is now easy to verify that

$$\begin{aligned}(Y^{\bullet})_j &= \{n : \langle n,j\rangle \in Y^{\bullet}\}\\ &= \{n : \langle n,j+1\rangle \in Y\}\\ &= \left\{n : \mathscr{A}(n,j,((Y)^{Vj+1})^{\bullet})\right\}\\ &= \left\{n : \mathscr{A}(n,j,(Y^{\bullet})^{Wj})\right\},\end{aligned}$$

hence $Y^{\bullet}$ is a hierarchy of the proper form. □

Proof of Theorem A.2. $\mathsf{oATR}_0^- \subseteq \mathsf{ATR}_0^-$ follows immediately by definition. For the converse direction we work in oATR_0^-. Let W be a well ordering and $\mathscr{A}(n,j,X)$ arithmetical with no other set variable occurring freely besides X. The goal is to prove the existence of a set Z satisfying $\mathcal{H}_{\mathscr{A}}(W,Z)$. By the last two lemmas we can assume without loss of generality that

$$\forall j \forall X \big(\{n : \mathscr{A}(n,j,X)\} \neq \emptyset \big), \tag{A.2}$$

and that the minimum of W, denoted $\min(W)$, is 0. Define the ordering $W^{\triangle}$ such that $\langle i_1,i_2\rangle \leq_{W^{\triangle}} \langle j_1,j_2\rangle$ iff

$$(i_2 \leq_W i_1 \wedge j_2 \leq_W j_1) \wedge \big((i_1 <_W j_1) \vee (i_1 = j_1 \wedge i_2 \leq_W j_2)\big).$$

Note that W is a well ordering, $\mathrm{field}(W^{\triangle}) = \{\langle j_1,j_2\rangle : j_2 \leq_W j_1\}$, and $\min(W^{\triangle}) = \langle 0,0\rangle$. Next, we let

$$\begin{aligned}\mathscr{A}^d(n,X) :\equiv \big(&\forall k \neg \mathscr{D}(k,X) \wedge \mathscr{A}(n,0,X^d)\big)\\ &\vee \big(\exists k(\mathscr{D}(k,X) \wedge \mathscr{A}(n,k,X^d))\big)\end{aligned}$$

where

$$\begin{aligned}\mathscr{D}(k,X) &:\equiv \exists m,l\big(\langle m,\langle k,l\rangle\rangle \in X\big) \wedge \forall m\big(\langle m,\langle k,k\rangle\rangle \notin X\big),\\ X^d &:= \{\langle m,l\rangle : \langle m,\langle l,l\rangle\rangle \in X\}\,.\end{aligned}$$

Applying (oATR^-) to $\mathscr{A}^d$ and $W^\triangle$ yields a set Y of the form

$$Y = \Big\{\langle n,j\rangle : j \in \mathrm{field}(W^\triangle) \wedge \mathscr{A}^d(n,(Y)^{W^\triangle j})\Big\}\,.$$

Consider $j \in \mathrm{field}(W^\triangle)$. If $\forall k \neg\mathscr{D}(k,(Y)^{W^\triangle j})$ holds, then by definition of $\mathscr{A}^d(n,X)$, it follows that

$$(Y)_j = \Big\{n : \mathscr{A}(n,0,((Y)^{W^\triangle j})^d)\Big\}\,,$$

hence by (A.2), $(Y)_j \neq \emptyset$. Otherwise, if k is such that $\mathscr{D}(k,(Y)^{W^\triangle j})$, then by (A.2) and definition of $\mathscr{A}^d(n,X)$ we obtain

$$\emptyset \neq \Big\{n : \mathscr{A}(n,k,((Y)^{W^\triangle j})^d)\Big\} \subseteq (Y)_j,$$

and so $(Y)_j \neq \emptyset$. Thus, we showed

$$(\forall j \in \mathrm{field}(W^\triangle))(Y)_j \neq \emptyset. \tag{A.3}$$

Next, we point out some crucial properties of $\mathscr{D}(n,X)$. Let $h \in \mathrm{field}(W)$. First, observe that $\forall k\neg\mathscr{D}(k,\emptyset)$. Moreover, if $k \notin \mathrm{field}(W)$ and $X \subseteq \mathrm{Nat} \times \mathrm{field}(W^\triangle)$, $\neg\mathscr{D}(k,X)$ follows. From (A.3) we can now deduce

$$\forall k \neg\mathscr{D}(k,(Y)^{W^\triangle \langle h,0\rangle}).$$

Moreover, if $h, i \neq 0$ and $i \leq_W h$ we get by (A.3) that

$$\forall k\big(\mathscr{D}(k,(Y)^{W^\triangle \langle h,i\rangle}) \leftrightarrow k = h\big).$$

The two properties above together with the definition of $\mathscr{A}^d(n,X)$ yield the following for all $h \in \mathrm{field}(W)$:

$$\mathscr{A}^d(n,(Y)^{W^\triangle \langle h,h\rangle}) \leftrightarrow \mathscr{A}(n,h,((Y)^{W^\triangle \langle h,h\rangle})^d).$$

Furthermore, a simple calculation shows that $((Y)^{W^{\triangle}\langle h,h\rangle})^d = (Y^d)^{Wh}$:

$$\begin{aligned}
((Y)^{W^{\triangle}\langle h,h\rangle})^d &= \left\{\langle n,l\rangle : \langle n,\langle l,l\rangle\rangle \in (Y)^{W^{\triangle}\langle h,h\rangle}\right\} \\
&= \{\langle n,l\rangle : \langle n,\langle l,l\rangle\rangle \in Y \wedge \langle l,l\rangle <_{W^{\triangle}} \langle h,h\rangle\} \\
&= \{\langle n,l\rangle : \langle n,\langle l,l\rangle\rangle \in Y \wedge l <_W h\} \\
&= \left\{\langle n,l\rangle : \langle n,l\rangle \in Y^d \wedge l <_W h\right\} \\
&= (Y^d)^{Wh}.
\end{aligned}$$

We now have all the ingredients to establish that Y^d is the desired set:

$$\begin{aligned}
(Y^d)_h &= \{n : \langle n,\langle h,h\rangle\rangle \in Y\} \\
&= \left\{n : \mathscr{A}^d(n,(Y)^{W^{\triangle}\langle h,h\rangle})\right\} \\
&= \left\{n : \mathscr{A}(n,h,((Y)^{W^{\triangle}\langle h,h\rangle})^d)\right\} \\
&= \left\{n : \mathscr{A}(n,h,(Y^d)^{Wh})\right\}.
\end{aligned}$$

This concludes the proof. $\square$

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Index

$\mathscr{A}^\star$, 69
$\mathscr{A}^\circ$, 56
$|\mathscr{A}|_2$, 70
$|\mathscr{A}|_{\mathrm{s}}$, 57
$\mathsf{ACA}_0, \mathsf{ACA}, (\mathrm{ACA})$, 13–14
ACA_0^+, 64
atomic formula
  of $\mathcal{L}_2$, 6
  of $\mathcal{L}_{\mathrm{s}}$, 50
$\mathsf{ATR}_0, \mathsf{ATR}, (\mathrm{ATR})$, 18
$\mathsf{ATR}_0^-, \mathsf{ATR}^-, (\mathrm{ATR}^-)$, 31
$\mathsf{ATR}_0^{\mathsf{set}}$, 96
axiom
  in $\mathcal{L}_2$
    logical, 8
    number-theoretic, 12
  in $\mathcal{L}_{\mathrm{s}}$
    Beta, 94
    Countability, 95
    Infinity, 95
    logical, 52
    number-theoretic, 54
    ontological, 54
    Pair, 55
    Rudimentary Closure, 55
    set-theoretic, 55
  non-logical, 10
Ax(PA), 54

$\mathsf{BETA}_0, \mathsf{BETA}$, 95
BS, 53
$\mathsf{BS}^0, \mathsf{BS}^1$, 55
BS^2, 93

$\mathrm{C}_{\mathrm{Iso}}(\langle \sigma, \tau \rangle, X, T)$, 60
class
  in $\mathcal{L}_2$, 8
  in $\mathcal{L}_{\mathrm{s}}$, 51
comprehension schema
  arithmetical, 13
  full, 13

$(\Delta_0\text{-Sep})$, 55
$(\Delta^1_1\text{-CA})$, 21
$(\Delta^1_2\text{-CA})$, 102
$\Delta^1_1\text{-}\mathsf{TR}_0, \Delta^1_1\text{-}\mathsf{TR}$,
    $(\Delta^1_1\text{-TR})$, 20
derivation, 10

$\widehat{\mathsf{EID}_1}$, 45
ε_0, 34
$\in^\star$, 60
equality
  $=_{\mathrm{N}}$, 49
  $=^\star$, 60
  of objects, 51
  of set variables, 8, 14

field

of a set variable, 16
of an object variable, 52
field variable, 17
finite sequence, 14
finite set
in $\mathcal{L}_2$, 15
in $\mathcal{L}_\mathrm{s}$, 94
$\mathrm{Fml}(\mathcal{L}_2)$, 7
$\mathrm{Fml}(\mathcal{L}_\mathrm{s})$, 50
formal system, 10
model of, 12
subsystem, 10
formula, *see also* atomic formula, literal
of $\mathcal{L}_2$, 6
arithmetical, 13
Π^1_k, 13
Π^1, 21
Σ^1_k, 13
Σ^1 , 21
X-positive, 7
of $\mathcal{L}_\mathrm{s}$, 50
Δ_0, 51
Π_k, 51
Π, 71
Σ_k, 51
Σ, 71
$\mathsf{FP}_0, \mathsf{FP}, (\mathrm{FP})$, 18
$\mathsf{FP}_0^-, \mathsf{FP}^-, (\mathrm{FP}^-)$, 37
$\mathsf{FP}^\ominus, (\mathrm{FP}^\ominus)$, 44
$\mathrm{FrVar}(\mathscr{A})$, 7

Γ_0, 34

$\mathcal{H}_\mathscr{A}(\alpha, Y)$, 36
$\mathcal{H}_\mathscr{A}(k, W, Y)$, 17
$\mathcal{H}^\cup_\mathscr{A}(k, Y)$, 64
$\tilde{\mathcal{H}}_\mathscr{F}(\alpha, Y)$, 42
hereditarily countable set, 94
hereditarily finite set, 94

$\widehat{\mathsf{ID}}_1, (\widehat{\mathsf{ID}}_1)$, 37
induction
axiom, 13
full, 14
$(\mathcal{L}_\mathrm{s}\text{-}\mathrm{I}_\in)$, 55
$(\Delta_0\text{-}\mathrm{I}_\mathrm{N})$, 55
$(\Delta_0\text{-}\mathrm{I}_\in)$, 93
$(\mathcal{L}_\mathrm{s}\text{-}\mathrm{I}_\mathrm{N})$, 54
inference rule
of $\mathcal{L}_2$, 9
$\mathrm{Iso}(X, T)$, 60

$\mathcal{L}(\mathcal{A})$, 37
$\mathcal{L}_1$, 6
$\mathcal{L}_1^{\mathsf{FP}}$, 37
$\mathcal{L}_2$, 5
$\mathcal{L}_2^{\mathsf{FP}}$, 45
$\mathcal{L}_\mathrm{s}$, 49
$\prec, \preceq$, 35
$\mathrm{lh}_T^{-1}(\tau)$, 64
literal
of $\mathcal{L}_2$, 6
of $\mathcal{L}_\mathrm{s}$, 50
$\mathrm{LO}(X)$, 16

$\mathsf{M}\Delta^1_1\text{-}\mathsf{FP}_0, \mathsf{M}\Delta^1_1\text{-}\mathsf{FP}$, $(\mathrm{M}\Delta^1_1\text{-FP})$, 19

N, 49
$\underline{\mathrm{n}}$, 49
$\mathbb{N}$, 14
Nat, 14
negation
in $\mathcal{L}_2$, 7
in $\mathcal{L}_\mathrm{s}$, 51

numeral, 6

$\mathsf{oATR}_0, \mathsf{oATR}$, (oATR), 107
$\mathsf{oATR}_0^-, \mathsf{oATR}^-$,
(oATR$^-$), 107–108
order type, 34
ordering
countable linear, 16
countabe well, 16
ordinal
finite, 94
limit, 94
proof-theoretic, 35
provable, 34
successor, 94

$P_{\mathscr{A}}$, 37
pairing map, 15
(Π^1_2-CA), 102
$\Pi^1_1\text{-}\mathsf{Red}_0, \Pi^1_1\text{-}\mathsf{Red}$, ($\Pi^1_1$-Red), 20
(Π^1_2-Red), 102
(Π^1-Red), 87
(Π-Red), 87
$\oplus$, 60
$\mathsf{pr\text{-}ATR}_0^-, \mathsf{pr\text{-}ATR}^-$,
(pr-ATR$^-$), 36
primitive recursive function and
relation symbol, 6
Prog$(\prec, X)$, 34
Prog(X), 35

Q, 5, 50

$\mathscr{R}(\alpha), \mathscr{R}(\alpha, X)$, 42
reflexive set, 16
Rep(T), 58
Rep$(\vec{U})$, 70

S, 50

$|(\text{Schem})|_2$, 83
second order arithmetic, 12
Seq, 15
(Σ^1_1-AC), 21
$\Sigma^1_1\text{-}\mathsf{AC}_0^+$, 70
$\Sigma^1_1\text{-}\mathsf{Red}_0, \Sigma^1_1\text{-}\mathsf{Red}$, ($\Sigma^1_1$-Red), 20
(Σ^1_2-Red), 102
(Σ^1-Red), 87
(Σ-Red), 87
Sp(X), 42
Sp$^*(\alpha, X)$, 42
structure
in $\mathcal{L}_2$, 11
in $\mathcal{L}_\mathrm{s}$, 53
substitution
of number variables, 7
of set variables, 7
subsystem, 10

term
class, 7
number, of $\mathcal{L}_2$, 6
of $\mathcal{L}_\mathrm{s}$, 50
ordinal, 35
TI(α, X), 35
TI$(\mathscr{A}, X)$, 99
TI$(\prec, X)$, 34
Tm$(\mathcal{L}_2)$, 6
Tm$(\mathcal{L}_\mathrm{s})$, 50
transitive set, 94
translation
from $\mathcal{L}_2$ to $\mathcal{L}_\mathrm{s}$, 55–57
from $\mathcal{L}_\mathrm{s}$ to $\mathcal{L}_2$, 69–70
tree
n-, 58
representation, 58

val$_t[y]$, 56

Veblen function
  $\varphi_{\cdot}(\cdot)$, 34
  $\hat{\varphi}_{\cdot}(\cdot)$, 36

ω, 14
well-founded set, 16
$\mathrm{WF}(X)$, 16
$\mathrm{WO}(\alpha)$, 35
$\mathrm{WO}(X)$, 16
$\mathsf{w}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0, \mathsf{w}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}$,
  $(\mathrm{w}\text{-}\Sigma^1_1\text{-TDC})$, 19

x°, X°, 56
$X =_{\mathrm{obj}} Y$, 73
$X^{\langle\sigma,\tau\rangle}$, 62

$(Y)_i, (Y)^{Wj}$, 17
Y^{-1}, 62

Z_2, 12–13

Erklärung

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